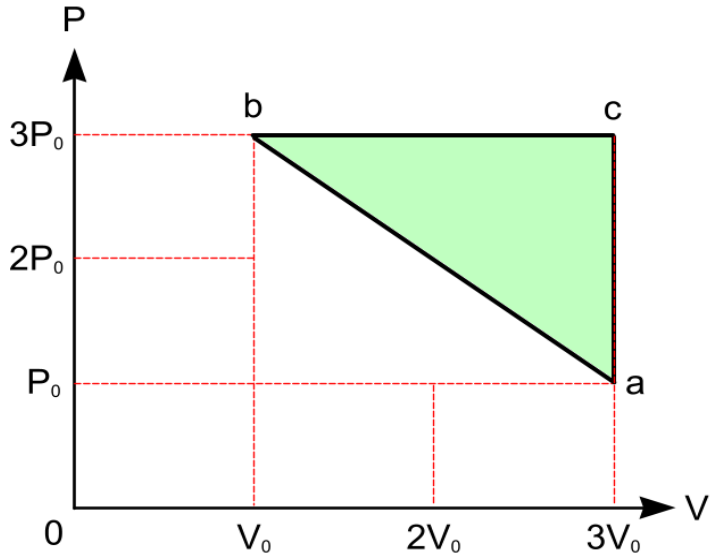
Theoretical Physics **Exam 2** **March 24, 2020**
Prof. Ruiz, UNCA **Closed Book/Notes but Calculator is Permitted**

1. Engine. Use $PV = nRT$, $P_0V_0 = nRT_0$, and take the number of moles to be such that $nR = 1\ \frac{\text{joules}}{\text{kelvins}} = 1\ \frac{J}{K}$. Take the temperature $T_0 = 300\text{K}$.



a. Show that $T_a = 900\text{K}$.

b. Show that $T_b = 900\text{K}$.

c. Show that $T_c = 2700\text{K}$.

d. Show that $\Delta Q_{a\to b} = -1200\text{ J}$

e. Show that $\Delta Q_{b\to c} = 4500\text{ J}$

f. Show that $\Delta Q_{c\to a} = -2700\text{ J}$.

g. Calculate $\eta = \frac{W_{\text{net}}}{Q_{\text{in}}}$.

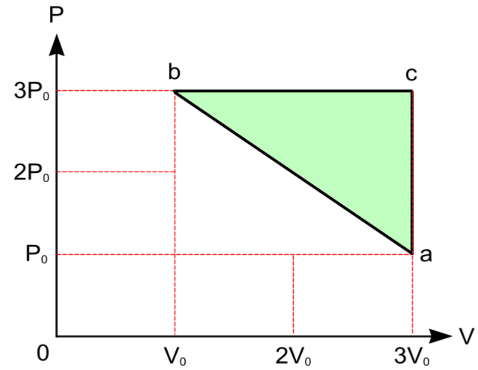
2. Stat Mechanics. The energy levels are 0, ε, 2ε, 3ε, 4ε, 5ε, …, with ε = kT. Use the identity $S = 1 + r + r^2 + r^3 + r^4 ... = \frac{1}{1-r}$, where r < 1, to show that the partition function $Z = \sum_{n=0}^{\infty} e^{-\frac{E_n}{kT}}$ is 1.58. Find the occupation probability in each of these levels: 0, ε, 2ε, 3ε, 4ε.

3. Quantum Mechanics. The time independent Schrödinger equation in one dimension can be written as $H\psi = E\psi$ where $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$. Let $V(x) = \frac{1}{2}m\omega^2 x^2$. Then consider special units where $\hbar = m = \omega = 1$. Show that $\psi(x) = \frac{1}{\sqrt{\pi}} e^{-\frac{x^2}{2}}$ is an eigenfunction, i.e., $H\psi = E\psi$, and find the eigenvalue E.

4. Eigenvectors and Eigenvalues. Find the normalized eigenvectors and eigenvalues for the matrix operator $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$. A vector $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ is normalized if $c_1^* c_1 + c_2^* c_2 = 1$.

Theoretical Physics, Doc Ruiz **Exam 2 - Solutions** **March 24, 2020**

1. Engine. Ideal gas with number of moles so that $PV = T$. Also, $P_0V_0 = T_0 = 300\text{K}$.



a. At a, $P_aV_a = T_a = P_0(3V_0) = 3T_0 = 900\text{K}$.

b. At b, $P_bV_b = T_b = (3P_0)V_0 = 3T_0 = 900\text{K}$.

c. At c, $P_cV_c = T_c = (3P_0)(3V_0) = 9T_0 = 2700\text{K}$.

d. $\Delta U_{ab} = 0$, $\Delta Q_{ab} = \Delta W_{ab} = -4P_0V_0 = -1200$ J , using Trapezoid Area: (average height $2P_0$)(base of $2V_0$).

e. $\Delta U_{bc} = \frac{3}{2}nR\Delta T_{bc} = \frac{3}{2}\Delta T_{bc} = \frac{3}{2}6T_0 = 9\cdot 300 = 2700$ J

$\Delta W_{bc} = 3P_0 2V_0 = 6P_0V_0 = 6T_0 = 1800$ J and $\Delta Q_{bc} = \Delta U_{bc} + \Delta W_{bc} = 2700 + 1800 = 4500$ J

f. $\Delta W_{ca} = 0$ so $\Delta Q_{ca} = \Delta U_{ca} = \frac{3}{2}\Delta T_{ca} = \frac{3}{2}(T_a - T_c) = \frac{3}{2}(900 - 2700) = -2700$ J

g. $\eta = \frac{W_{\text{net}}}{Q_{\text{in}}} = \frac{W_{\text{enclosed area}}}{Q_{\text{in}}} = \frac{(2P_0)(2V_0)/2}{Q_{bc}} = \frac{2T_0}{4500} = \frac{600}{4500} = \frac{6}{45} = \frac{2}{15}$

2. Stat Mech. The energy levels are 0, ε, 2ε, 3ε, 4ε, 5ε, …, with ε = kT. Let $r = e^{-1}$.

$$Z = \sum_{n=0}^{\infty} e^{-\frac{E_n}{kT}} = e^0 + e^{-1} + e^{-2} + e^{-3} ... = 1 + r + r^2 + r^3 + ... = \frac{1}{1-r} = \frac{1}{1-e^{-1}} = \frac{e}{e-1} = 1.58$$

$\frac{n_0}{N} = \frac{e^0}{Z} = \frac{1}{1.58} = 0.63$, $\frac{n_1}{N} = \frac{e^{-1}}{1.58} = 0.23$, $\frac{n_2}{N} = \frac{e^{-2}}{1.58} = 0.09$, $\frac{n_3}{N} = \frac{e^{-3}}{1.58} = 0.03$, $\frac{n_4}{N} = \frac{e^{-4}}{1.58} = 0.01$

3. Quantum Mech. $H = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2$ and $\psi(x) = \frac{1}{\sqrt{\pi}}e^{-\frac{x^2}{2}}$. Show $H\psi = E\psi$. Find E .

$$-\frac{1}{2}\frac{d^2\psi}{dx^2} = \left[-\frac{1}{2}\frac{d^2}{dx^2}\right]\frac{1}{\sqrt{\pi}}e^{-\frac{x^2}{2}} = \frac{1}{2}\frac{d}{dx}\left[x\frac{1}{\sqrt{\pi}}e^{-\frac{x^2}{2}}\right] = \frac{1}{2}\left[\frac{1}{\sqrt{\pi}}e^{-\frac{x^2}{2}}\right] - \frac{1}{2}\left[\frac{x^2}{\sqrt{\pi}}e^{-\frac{x^2}{2}}\right]$$

$-\frac{1}{2}\frac{d^2\psi}{dx^2} = \frac{1}{2}\psi - \frac{1}{2}x^2\psi$, $-\frac{1}{2}\frac{d^2\psi}{dx^2} + \frac{1}{2}x^2\psi = \frac{1}{2}\psi$. Eigenfunction ψ , eigenvalue $\frac{1}{2}$.

4. Eigenvectors and Eigenvalues. $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$, $\det\begin{bmatrix} 1-\lambda & 2 \\ 3 & -\lambda \end{bmatrix} = 0$, $\lambda^2 - \lambda - 6 = 0$

$(\lambda - 3)(\lambda + 2) = 0$ gives eigenvalues 3 and -2.

$\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 3\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ gives $c_1 + 2c_2 = 3c_1$, $c_2 = c_1$, normalized eigenvector $u = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = -2\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ gives $c_1 + 2c_2 = -2c_1$, $c_2 = -\frac{3}{2}c_1$ leading to $v = \frac{1}{\sqrt{13}}\begin{bmatrix} 2 \\ -3 \end{bmatrix}$.