# Class M. The Method of Frobenius

M1. Method of Frobenius $\quad Y = \sum_{k=0}^{\infty} a_k x^k$

M2. Differential Equations in Physics

M3. The Legendre Differential Equation

$P = l(l+1)$

$l = 0, 1, 2, 3 \ldots$

Can let $m = k-2$

$k = m+2$

$k-1 = m+1$

$k-2 = m$

Then replace $m$ with a relabel of $k$.

Finally check that you can still start at 0.

$$(1-x^2)y'' - 2xy' + Py = 0 \qquad y' \equiv \frac{dy}{dx} \quad y'' \equiv \frac{d^2y}{dx^2}$$

Step 1. "Series Plug In"

$$y = \sum_{k=0}^{\infty} a_k x^k \qquad y' = \sum_{k=0}^{\infty} k a_k x^{k-1} \qquad y'' = \sum_{k=0}^{\infty} k(k-1) a_k x^{k-2}$$

$$(1-x^2)\sum_{k=0}^{\infty} k(k-1)a_k x^{k-2} - 2x\sum_{k=0}^{\infty} k a_k x^{k-1} + p\sum_{k=0}^{\infty} a_k x^k = 0$$

No harm in keeping $k=0$ & $k=1$ since they give zero.

$$\sum_{k=0}^{\infty} k(k-1)a_k x^{k-2} - \sum_{k=0}^{\infty} k(k-1)a_k x^k - 2\sum_{k=0}^{\infty} k a_k x^k + p\sum_{k=0}^{\infty} a_k x^k = 0$$

really starts at $k=2$

Same Same Same

Step 2. "Fix the Exponents"

Same result

$$\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k - \sum_{k=0}^{\infty} k(k-1)a_k x^k - 2\sum_{k=0}^{\infty} k a_k x^k + p\sum_{k=0}^{\infty} a_k x^k = 0$$

Step 3. "The Arbitrary Trick"

$$\sum_{k=0}^{\infty}\Big[\underbrace{(k+2)(k+1)a_{k+2} - k(k-1)a_k - 2k a_k + p a_k}_{\text{zero}}\Big]x^k = 0$$

True for all $x$. Can consider $x$ arbitrary.

$$(k+2)(k+1)a_{k+2} = \Big[\underbrace{k(k-1) + 2k}_{k(k+1)} - p\Big]a_k$$

Recursive, Recursion

Step 4. "Recurrence Relation"

$p = l(l+1)$

Series must not run wild. Must terminate to be finite.

$$a_{k+2} = \left[\frac{k(k+1) - P}{(k+1)(k+2)}\right]a_k \qquad k = 0, 1, 2, \ldots$$

$k_{max}$ will be $\quad K_{max}(K_{max}+1) - P = 0 \quad$ for some $k_{max}$

Some positive integer $l$. $\quad P = l(l+1) \quad l = 0, 1, 2 \ldots$ to make it happen.

The $l$ corresponds to the angular momentum quantum number in the hydrogen atom. $L^2 = \hbar^2 l(l+1)$

## M4. Legendre Polynomials

$0^{th}$ Legendre Polynomial $(l=0)$ $\Rightarrow p = l(l+1) = 0$

Choose $a_1 = 0$ to keep the odd cases from going wild.

$$a_{k+2} = \frac{k(k+1)}{(k+1)(k+2)} a_k$$

$$k=0 \quad a_2 = \frac{0(0+1)}{(0+1)(0+2)} a_0 = 0$$

Convention - choose $a_0$ so $P_l(1) = 1$

$P_0(x) = a_0$, $a_0 = 1$

For $l$ even, the evens will terminate (self destruct) at some point. When $k=l$ even $l$.

Same goes for odds when $l$ is odd. Then we take $a_0$ to be zero.

$1^{st}$ Legendre Polynomial $(l=1)$ $\quad l(l+1) = 1(1+1) = 2$

$$a_{k+2} = \left[\frac{k(k+1) - l(l+1)}{(k+1)(k+2)}\right] a_k = \left[\frac{k(k+1) - 2}{(k+1)(k+2)}\right] a_k$$

Force $a_0 = 0$ because $k(k+1) - l(l+1) = 0$ for $k_{max}$ where $k_{max}$ is odd, but evens will not automatically stop.

Take $k=1$ zero

$$a_3 = \left[\frac{1(1+1) - 2}{(1+1)(1+2)}\right] a_1 = 0$$

$\Rightarrow a_1 = 1$

$P_1(x) = a_1 x^1 = a_1 x \quad P_1(1) = 0$

$P_1(x) = x$

$2^{nd}$ Legendre Polynomial $(l=2)$

$a_1 = 0$

$$a_{k+2} = \left[\frac{k(k+1) - 2(2+1)}{(k+1)(k+2)}\right] a_k = \left[\frac{k(k+1) - 6}{(k+1)(k+2)}\right] a_k$$

$$k=0 \quad a_2 = \left[\frac{0(0+1) - 6}{(0+1)(0+2)}\right] a_0 = \frac{0-6}{(1)(2)} a_0 = -3a_0$$

for $k=2$ $\quad a_4 = 0$ since $k$ then is $k_{max} = 2$

$$P_2(x) = a_0 + a_2 x^2 = a_0 - 3a_0 x^2 = a_0(1 - 3x^2)$$

$$P_2(1) = 1 \Rightarrow a_0(1-3) = 1 \Rightarrow a_0 = -\frac{1}{2}$$

$$P_2(x) = -\frac{1}{2}(1 - 3x^2) = \frac{1}{2}(3x^2 - 1)$$

Summary: $\quad P_0(x) = 1 \quad P_1(x) = x \quad P_2(x) = \frac{1}{2}(3x^2 - 1)$

If you continue $\quad P_3(x) = \frac{1}{2}(5x^3 - 3x)$