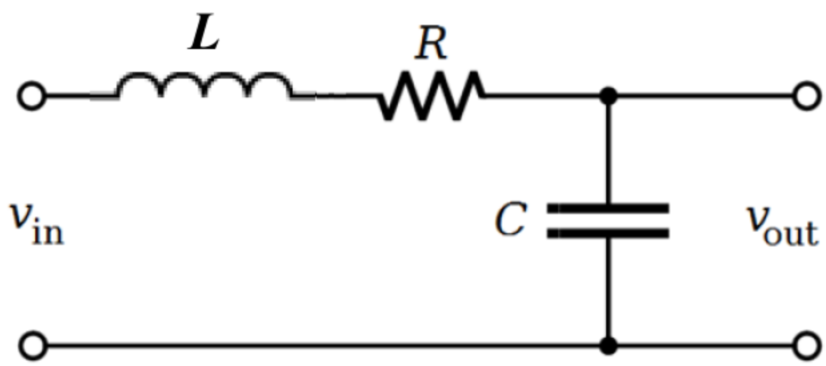
Theoretical Physics
Prof. Ruiz, UNC Asheville
Chapter V Homework. Transfer Functions

V1. The LRC Circuit. Find the transfer function $H(\omega)$ for the LRC circuit shown below.



Your impedances are $Z_L = j\omega L$, $Z_R = R$, and $Z_C = \dfrac{1}{j\omega C}$, where $j = \sqrt{-1}$. From your transfer function $H(\omega)$ find the relative transmission $|H(\omega)|$. Show that the charge $q(t)$ on the capacitor for an input voltage $V_{in}(t) = V_0 \sin \omega t$ is given by

$$q(t) = \frac{V_0}{L} \frac{\sin(\omega t + \phi)}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4\beta^2\omega^2}} \quad \text{with} \quad \tan\phi = \frac{2\beta\omega}{\omega^2 - \omega_0^2},$$

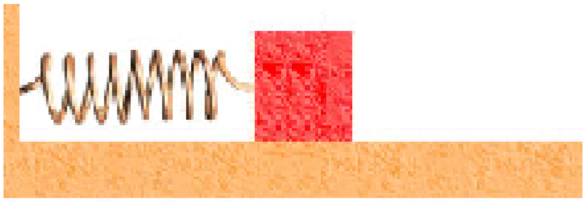
$$\text{where} \quad \omega_0 = \frac{1}{\sqrt{LC}} \quad \text{and} \quad \beta = \frac{R}{2L}.$$

Sketch $A(\omega)$ that appears in $q(t) = A(\omega)\sin(\omega t + \phi)$ and also sketch

$$\phi = \tan^{-1}\frac{2\beta\omega}{\omega^2 - \omega_0^2}.$$

Your sketch must be qualitatively correct and labeled appropriately for full credit.

V2. Resonances.



Courtesy David M. Harrison
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The differential equation for the driven harmonic oscillator is identical in terms of its basic math when compared to the LRC circuit you just analyzed.

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=F_0\sin\omega t$$

$$L\frac{d^2q}{dt^2}+R\frac{dx}{dt}+\frac{q}{C}=V_0\sin\omega t$$

Give the response $x(t)$ using the analogous nature of these equations and adapting your $q(t)$ from the previous problem. What are ω_0 and β in the mechanical analog to your electrical circuit?

Both the electrical and mechanical systems are resonance systems. Show that the maximum amplitude response occurs at $\omega_R=\sqrt{\omega_0^2-2\beta^2}$, which is called the resonance frequency (for the position or charge amplitude). Hint: You set $\frac{dA(\omega)}{d\omega}=0$ to do this max-min problem but you can take a shortcut since there is no ω in the numerator. In this case, you set what is in the square root of the denominator to zero since when your denominator is at a minimum, you have a maximum for your amplitude parameter. So you solve for the maximum response of the amplitude of oscillation or the charge on the capacitor. Find the maximum for the current amplitude (electrical case) or velocity amplitude for the oscillation (mechanical case). It will NOT occur at ω_R. Hint for Solution: The voltage across the resistor is in sync with the input voltage and your voltage across the resistor is proportional to the current since $V_R=IR$. So redo the transfer part in V1 by taking the voltage across the resistor as your output. You can then find the max for the current amplitude directly that way much faster. The longer way is to get the current from the derivative of the charge function using the capacitor as the output and then do a max-min on that.