Theoretical Physics
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Chapter T Homework - Solutions. Poles and the Residue Theorem

T1. An Integration Along the x-axis.

Use residues to evaluate $I = \int_{-\infty}^{\infty} \frac{e^{imx}dx}{x^2 - 3ix - 2}$ where $m > 0$.

Step 1. Change to Complex Plane and Find Your Poles

$$I = \oint \frac{e^{imz}}{z^2 - 3iz - 2} dz \quad \text{or} \quad I = \oint \frac{e^{imz}}{z^2 - 3iz - 2} dz \quad ???$$

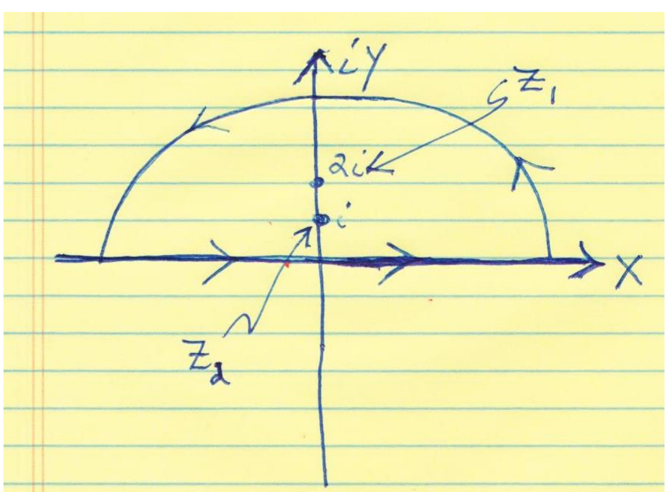
For now, don't worry about the semicircle part. Go and find the poles.

$$z^2 - 3iz - 2 = 0 \quad => \quad \frac{-(-3i) \pm \sqrt{(-3i)^2 - 4(1)(-2)}}{2(1)}$$

$$\frac{3i \pm \sqrt{-9+8}}{2} \quad => \quad \frac{3i \pm i}{2} \quad => \quad z_1 = 2i \quad \text{and} \quad z_2 = i$$

Step 2. Now. Where to Close the Path? We need to close in the Upper Plane so that

$$\lim_{y\to\infty} e^{imz} = \lim_{y\to\infty} e^{im(x+iy)} = e^{imx} \lim_{y\to\infty} e^{-my} = 0 \quad .$$



We want $I = \oint \frac{e^{imz}dz}{z^2 - 3iz - 2}$.

$$I = \oint F(z)dz$$

$$F(z) = \frac{e^{imz}}{z^2 - 3iz - 2}$$

A more formal proof that this upper circular portion vanishes is not necessary. The fact that for $m > 0$ the upper contour semicircle vanishes is called Jordan's Lemma. For the particular case $m < 0$, you would have to close in the lower half plane since that is the semicircle that vanishes in the latter case. So what you do in practice is the following. You check to see if see $e^{imz} \to 0$ as $y \to \infty$ (closing in the upper plane) or $y \to -\infty$ (closing in the lower plane) and pick your semicircle appropriately.

The DEAL. In practice, for all future homeworks and exams, unless you are specifically requested for more detail, you simply pick the appropriate contour and go on to do your residues.

Step 3. Sum Your Residues

$$I = 2\pi i \sum_n Res(F, z_n) \quad \text{with} \quad F(z) = \frac{e^{imz}}{(z-z_1)(z-z_2)}$$

$$z_1 = 2i \quad \text{and} \quad z_2 = i$$

$$I = 2\pi i\left[Res(F, z_1) + Res(F, z_2)\right]$$

$$I = 2\pi i\left[\frac{e^{imz_1}}{(z_1 - z_2)} + \frac{e^{imz_2}}{(z_2 - z_1)}\right]$$

$$I = \frac{2\pi i}{z_1 - z_2}\left[e^{imz_1} - e^{imz_2}\right] \quad \Rightarrow \quad I = \frac{2\pi i}{i}\left[e^{-2m} - e^{-m}\right]$$

$$I = 2\pi\left[e^{-2m} - e^{-m}\right] \quad \Rightarrow \quad I = 2\pi e^{-m}\left[e^{-m} - 1\right]$$

$$\boxed{I = 2\pi e^{-1}\left[e^{-1} - 1\right]}$$

$$\boxed{I = \frac{2\pi}{e}\left[\frac{1}{e} - 1\right] = -1.46}$$

T2. An Angle Integration.

Consider the integral $I = \int_0^{2\pi} \frac{d\theta}{5+3\cos\theta}$.

Let $z = Re^{i\theta}$ where $R = 1$ so that you have an integral in the complex plane with unit radius. Get everything in terms of z. Then use the residue theorem to obtain your answer.

$$z = e^{i\theta}$$

$$dz = ie^{i\theta}d\theta = izd\theta$$

$$d\theta = \frac{dz}{iz} \qquad \cos\theta = \frac{1}{2}\left[e^{i\theta} + e^{-i\theta}\right] = \frac{1}{2}\left[z + \frac{1}{z}\right]$$

$$I = \oint \frac{\frac{1}{iz}dz}{5 + \frac{3}{2}\left[z + \frac{1}{z}\right]}$$

$$I = \frac{1}{i}\oint \frac{dz}{5z + \frac{3}{2}\left[z^2 + 1\right]}$$

$$I = \frac{2}{i}\oint \frac{dz}{3z^2 + 10z + 3}$$

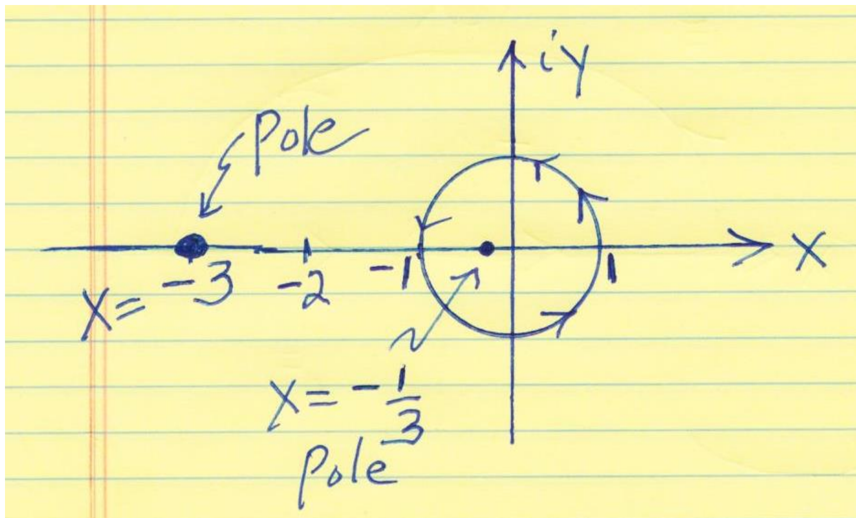
Solution. Start with $3z^2 + 10z + 3 = 0$. Use the quadratic equation.

$$\frac{-10 \pm \sqrt{10^2 - 4(3)(3)}}{2(3)} \quad \Rightarrow \quad \frac{-10 \pm \sqrt{100-36}}{6}$$

$$\frac{-10 \pm \sqrt{64}}{6}$$

$$\frac{-10 \pm 8}{6}$$

$-\frac{1}{3}$ and -3



Since our radius is 1, we only need the -1/3 pole.

$$I = 2\pi i \sum_n Res(F, z_n) \quad \text{with} \quad F(z) = \frac{2}{i}\frac{1}{3(z-z_1)(z-z_2)}$$

$$z_1 = -\frac{1}{3} \quad \text{and} \quad z_2 = -3$$

$$I = 2\pi i Res(F, z_1) \qquad I = 2\pi i \frac{2}{i}\frac{1}{3(z_1 - z_2)} = \frac{4\pi}{3}\frac{1}{\left[-\frac{1}{3}+3\right]}$$

$$I = \frac{4\pi}{3}\frac{1}{\left[-\frac{1}{3}+3\right]} = \frac{4\pi}{3}\frac{1}{8/3} = \frac{4\pi}{3}\frac{3}{8} = \frac{\pi}{2} \qquad \boxed{I = \frac{\pi}{2}}$$