Theoretical Physics
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Chapter S Homework - Solutions. Cauchy Integral Formula

S1. Analytic Functions. Take $f(z) = z^n$, where $n \neq -1$ so we avoid $f(z) = \frac{1}{z}$. Your goal is to show that $f(z) = z^n$ is analytic, i.e., the Cauchy-Riemann conditions hold. Then we are in good shape since this covers any function of the form $g(z) = \sum_{n=0}^{\infty} c_n z^n$. That means functions like the cosine and sine are in. The hint below leads you to the solution.

Hint. Let your $z = x + iy$ be represented as

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta} .$$

Now apply your n[th] power to the z. Show that your u and v in $f(z) = z^n = u + iv$ are

$$u = r^n \cos(n\theta) \text{ and } v = r^n \sin(n\theta) .$$

The $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$ part is de Moivre's formula.

Finally, use the chain rule to calculate your partial derivations. As an example, here is one of the four partial derivatives from the Cauchy-Riemann relations.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

For full credit, show the calculation of all partial derivatives from first principles with your results eventually expressed in terms of the polar coordinates r and θ.

Then show that the Cauchy-Riemann relations are satisfied.

Conclusion: Any function that can be expressed as a power series is analytic!

Solution.

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$$f(z) = z^n = r^n e^{in\theta}$$

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

$u = r^n \cos(n\theta)$ and $v = r^n \sin(n\theta)$.

We want to check to see if the Cauchy-Riemann conditions.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

So we will need to calculate the following.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x}$$

Here are the individual pieces we will need.

$\frac{\partial u}{\partial r}$	$\frac{\partial v}{\partial r}$	$\frac{\partial u}{\partial \theta}$	$\frac{\partial v}{\partial \theta}$
$\frac{\partial r}{\partial x}$	$\frac{\partial \theta}{\partial x}$	$\frac{\partial r}{\partial y}$	$\frac{\partial \theta}{\partial y}$

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r^2 = x^2 + y^2 \qquad \theta = \tan^{-1}\frac{y}{x}$$

Start with $r^2 = x^2 + y^2$ and use implicit differentiation:

$2rdr = 2xdx + 2ydy$ giving $\dfrac{\partial r}{\partial x} = \dfrac{x}{r} = \cos\theta$ and $\dfrac{\partial r}{\partial y} = \dfrac{y}{r} = \sin\theta$.

Below, we will need $\dfrac{d}{ds}\tan^{-1}s = \dfrac{1}{1+s^2}$, where $s = \dfrac{y}{x}$.

$$\frac{\partial\theta}{\partial x} = \frac{\partial}{\partial x}\tan^{-1}\frac{y}{x} = \frac{1}{1+(y/x)^2}\frac{\partial}{\partial x}\left[\frac{y}{x}\right] = \frac{1}{1+(y/x)^2}\left[-\frac{y}{x^2}\right]$$

$$\frac{\partial\theta}{\partial x} = -\frac{y}{x^2+y^2} = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r}$$

$$\frac{\partial\theta}{\partial y} = \frac{\partial}{\partial y}\tan^{-1}\frac{y}{x} = \frac{1}{1+(y/x)^2}\frac{\partial}{\partial y}\left[\frac{y}{x}\right] = \frac{1}{1+(y/x)^2}\left[\frac{1}{x}\right] = \frac{x}{x^2+y^2}$$

$$\frac{\partial\theta}{\partial y} = \frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}$$

Summary: $u = r^n\cos(n\theta)$ and $v = r^n\sin(n\theta)$.

$$\boxed{\frac{\partial r}{\partial x} = \cos\theta} \qquad \boxed{\frac{\partial r}{\partial y} = \sin\theta} \qquad \boxed{\frac{\partial\theta}{\partial x} = -\frac{\sin\theta}{r}} \qquad \boxed{\frac{\partial\theta}{\partial y} = \frac{\cos\theta}{r}}$$

We will also need the following four partial derivatives.

$$\boxed{\frac{\partial u}{\partial r} = nr^{n-1}\cos(n\theta)} \qquad \boxed{\frac{\partial u}{\partial\theta} = -nr^n\sin(n\theta)}$$

$$\boxed{\frac{\partial v}{\partial r} = nr^{n-1}\sin(n\theta)} \qquad \boxed{\frac{\partial v}{\partial\theta} = nr^n\cos(n\theta)}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$\frac{\partial u}{\partial x} = [nr^{n-1}\cos(n\theta)]\cos\theta + [-nr^{n}\sin(n\theta)]\left[-\frac{\sin\theta}{r}\right]$$

$$\boxed{\frac{\partial u}{\partial x} = nr^{n-1}\left[\cos(n\theta)\cos\theta + \sin(n\theta)\sin\theta\right]}$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$\frac{\partial v}{\partial y} = [nr^{n-1}\sin(n\theta)]\sin\theta + [nr^{n}\cos(n\theta)]\left[\frac{\cos\theta}{r}\right]$$

$$\boxed{\frac{\partial v}{\partial y} = nr^{n-1}\left[\sin(n\theta)\sin\theta + \cos(n\theta)\cos\theta\right]}$$

$$\boxed{\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}}$$

Now we check the other Cauchy-Riemann equation. Does $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$?

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$\boxed{\frac{\partial u}{\partial r} = nr^{n-1}\cos(n\theta)} \quad \boxed{\frac{\partial r}{\partial y} = \sin\theta} \quad \boxed{\frac{\partial u}{\partial \theta} = -nr^{n}\sin(n\theta)} \quad \boxed{\frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}}$$

$$\frac{\partial u}{\partial y} = [nr^{n-1}\cos(n\theta)\sin\theta] - nr^{n}\sin(n\theta)\left[\frac{\cos\theta}{r}\right]$$

$$\boxed{\frac{\partial u}{\partial y} = nr^{n-1}\left[\cos(n\theta)\sin\theta - \sin(n\theta)\cos\theta\right]}$$

$$\frac{\partial v}{\partial x} = nr^{n-1}\sin(n\theta)\cos\theta + nr^{n}\cos(n\theta)\left[\frac{-\sin\theta}{r}\right]$$

$$\boxed{\frac{\partial v}{\partial x} = nr^{n-1}\left[\sin(n\theta)\cos\theta - \cos(n\theta)\sin\theta\right]}$$

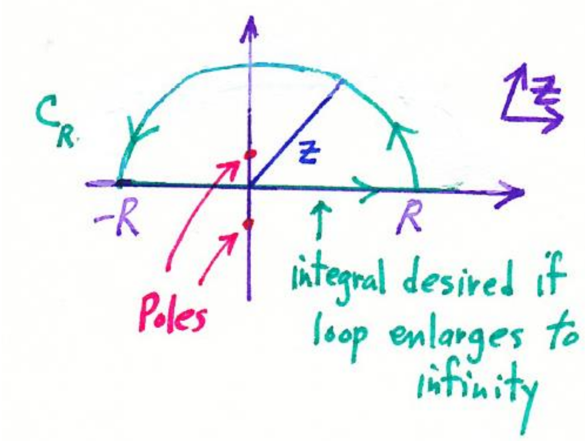
$$\boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

S2. Contour Integration. You will evaluate the integral $I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ using complex variable techniques. But first, evaluate this integral from the observation that

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}.$$

Show that you get the cute answer $I = \pi$.

Now you will do the same integral using the method of contour integration.



See two singularities marked in the figure, labeled as poles. These are at $z = \pm i$. Note that the path shown has an integration component along the real axis.

Use the Cauchy Integral formula to show that the integration along the closed contour indicated in the figure gives

$$I = \oint \frac{1}{1+z^2}\, dz = \pi$$

All we have to do now is let $R \to \infty$ and hope that the semicircle integration along C_R goes to zero. Note that then the complete enclosed contour integral gives a nonzero answer for just the path along the complete x-axis. You expect the semicircular path to give zero because you have your $I = \pi$, which you know is the answer.

Show

$$\lim_{R\to\infty} I_{C_R} = 0 \text{, where } I_{C_R} = \int_{C_R} \frac{1}{1+z^2} dz .$$

Hint. Let $z = Re^{i\theta}$ and express the integral in terms of R and θ. Then proceed.

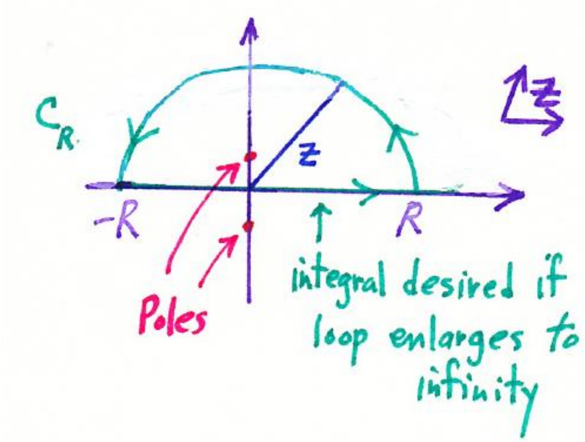
Solution.

Find $I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$. From the given, we know $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$.

Part 1. Evaluate the integral from the above given formua involving the derivative.

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{\infty} \frac{d}{dx}\tan^{-1} x dx = \tan^{-1} x\Big|_{-\infty}^{\infty} = \frac{\pi}{2} - \left[-\frac{\pi}{2}\right] = \pi$$

Part 2. Contour Integration.



$$I = \oint \frac{1}{1+z^2} dz$$

$$\frac{1}{1+z^2} = \frac{1}{(z+i)(z-i)}$$

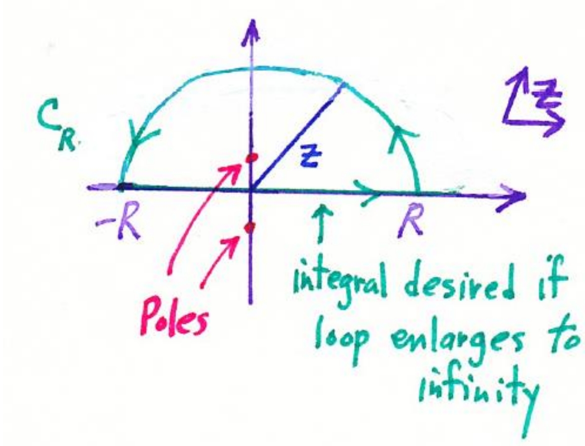
Let $F(z) = \frac{1}{z+i}$.

$$I = \oint \frac{1}{(z+i)(z-i)} dz = \oint \frac{F(z)}{z-i} = 2\pi i F(i)$$

$$I = 2\pi i \frac{1}{2i} = \pi$$

Part 3. Show

$\lim\limits_{R\to\infty} I_{C_R} = 0$, where $I_{C_R} = \int\limits_{C_R} \frac{1}{1+z^2} dz$.



Start with the Hint given that $z = Re^{i\theta}$.
Therefore, $dz = iRe^{i\theta}d\theta$.

$$I_{C_R} = \int_{C_R} \frac{1}{1+z^2}dz$$

$$I_{C_R} = \int_{C_R} \frac{1}{1+(i\,\mathrm{R}\mathrm{e}^{i\theta})^2} iRe^{i\theta}d\theta$$

$$I_{C_R} = \int_{C_R} \frac{1}{1-\mathrm{R}^2\mathrm{e}^{2i\theta}} iRe^{i\theta}d\theta$$

For R large $$I_{C_R} = \lim_{R\to\infty} \int_{C_R} \frac{1}{1-\mathrm{R}^2\mathrm{e}^{2i\theta}} iRe^{i\theta}d\theta.$$

$$I_{C_R} = \lim_{R\to\infty} \int_{C_R} \frac{1}{(-\mathrm{R}^2\mathrm{e}^{2i\theta})} iRe^{i\theta}d\theta = -\lim_{R\to\infty} \int_{C_R} \frac{\mathrm{e}^{-2i\theta}}{\mathrm{R}^2} iRe^{i\theta}d\theta$$

$$I_{C_R} = -i\lim_{R\to\infty} \int_{C_R} \frac{\mathrm{e}^{-i\theta}}{\mathrm{R}} d\theta = -i\lim_{R\to\infty} \frac{1}{R}\int_{C_R} \mathrm{e}^{-i\theta}\,d\theta$$

$$I_{C_R} = -i\lim_{R\to\infty} \frac{1}{R}\frac{\mathrm{e}^{-i\theta}}{(-i)}\bigg|_0^{\pi} = \lim_{R\to\infty} \frac{1}{R}\mathrm{e}^{-i\theta}\Big|_0^{\pi} = \lim_{R\to\infty} \frac{1}{R}[e^{-i\pi} - e^0]$$

$$I_{C_R} = \lim_{R\to\infty} \frac{1}{R}[(-1)-1] = \lim_{R\to\infty} \frac{(-2)}{R} = 0$$