Theoretical Physics
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Chapter R Homework - Solutions. Convolution

R1. Convolution. You will work with the following two functions.

$$f(t) = t \quad \text{and} \quad g(t) = t^2$$

a. Find the convolution

$$f(t) * g(t) = \int_0^t f(u) g(t-u) du \ .$$

b. Show that if you calculate the convolution by the following formula, you get the same result. Since this is true in general, this means that the convolution operation is commutative.

$$g(t) * f(t) = \int_0^t g(u) f(t-u) du$$

c. Show for this case that the Laplace transform of the convolution is given by the product of the Laplace transform of each function. When you do this part, you can look up the Laplace transforms in our "Laplace Transform Table."

$$L\{g(t) * f(t)\} = L\{g(t)\} L\{f(t)\} = G(s) F(s)$$

Solution.

Part a.

$$f(t) = t \quad \text{and} \quad g(t) = t^2$$

$$f(t) * g(t) = \int_0^t f(u) g(t-u) du = \int_0^t u(t-u)^2 du$$

$$f(t) * g(t) = \int_0^t u(t^2 - 2tu + u^2) du$$

$$f(t) * g(t) = \int_0^t (ut^2 - 2tu^2 + u^3) du$$

$$f(t) * g(t) = \left(\frac{u^2}{2} t^2 - 2t\frac{u^3}{3} + \frac{u^4}{4}\right)\Bigg|_0^t$$

$$f(t)*g(t)=\frac{t^4}{2}-\frac{2}{3}t^4+\frac{t^4}{4}$$

$$f(t)*g(t)=t^4\left[\frac{1}{2}-\frac{1}{3}+\frac{1}{4}\right]$$

$$f(t)*g(t)=t^4\left[\frac{6-8+3}{12}\right]=t^4\,\frac{1}{12}$$

$$\boxed{f(t)*g(t)=\frac{t^4}{12}}$$

Part b.

$$f(t)=t \quad \text{and} \quad g(t)=t^2$$

$$g(t)*f(t)=\int_0^t g(u)\,f(t-u)du=\int_0^t u^2(t-u)du$$

$$g(t)*f(t)=\int_0^t (u^2t-u^3)du$$

$$g(t)*f(t)=\left.(\frac{u^3}{3}t-\frac{u^4}{4})\right|_0^t$$

$$g(t)*f(t)=(\frac{t^4}{3}-\frac{t^4}{4})$$

$$\boxed{g(t)*f(t)=\frac{t^4}{12}}$$

Part c. We will use

$$L\{t^n\}=\frac{n!}{s^{n+1}}$$

$$f(t)=t \quad \text{and} \quad g(t)=t^2$$

$$g(t) * f(t) = \frac{t^4}{12}$$ from Parts (a) and (b).

Now check this. $L\{f(t) * g(t)\} = L\{\frac{t^4}{12}\} = \frac{1}{12}\frac{4!}{s^5} = \frac{1}{12}\frac{24}{s^5} = \frac{2}{s^5}$.

But note we get the same with $L\{f(t)\}L\{g(t)\} = L\{t\}L\{t^2\} = \frac{1}{s^2}\frac{2}{s^3} = \frac{2}{s^5}$.

R2. Dumping Radioactive Decay.

There is a pristine dumping ground totally free of any kind of trash. Then in comes the dump trucks dumping radioactive waste according to the dumping function $d(t)$. The radioactive-decay differential equation must now be modified to include dumping.

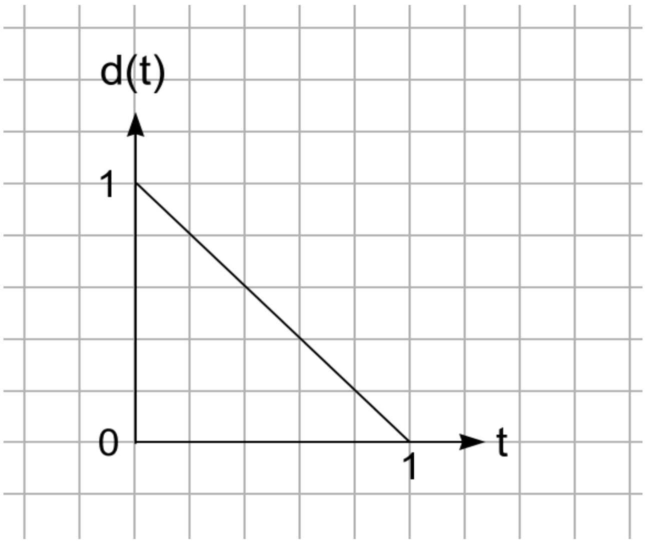
$$\frac{dn(t)}{dt} = -\lambda n(t) + d(t)$$

Your first term on the right is your radioactive-loss part due to decay. That piece is still proportional to $n(t)$. But you have to add the gain in radioactive particles to get your total rate of change $\frac{dn(t)}{dt}$. Show that the Laplace transform of your differential equation leads to your solution in the form $N(s) = F(s)D(s)$ where $L\{n(t)\} = N(s)$, $L\{d(t)\} = D(s)$, and $F(s)$ is everything else. What is $f(t)$? Give the solution $n(t)$ in the form of a convolution. Now consider a dumping function $d(t) = 1 - t$ for $0 \le t \le 1$ and $d(t) = 0$ elsewhere. Sketch this dumping function. Show that the "derivative trick" can be used to express your solution for this particular dumping function for times $t \ge 1$ as

$$n(t) = e^{-\lambda t}\left[1 - \frac{d}{d\lambda}\right]\left[\frac{1}{\lambda}(e^{\lambda} - 1)\right].$$

Finally evaluate the above and express $n(t)$ in simplest terms.

Solution. First, the “Sketch” is below.



$$\frac{dn(t)}{dt} = -\lambda n(t) + d(t)$$

$$L\{\frac{dn(t)}{dt}\} = L\{-\lambda n(t) + d(t)\}$$

$$L\{\frac{dn(t)}{dt}\} = -\lambda L\{n(t)\} + L\{d(t)\}$$

$$sN(s) - n(0) = -\lambda N(s) + D(s)$$

Note that $n(0)$ since there is no radioactive waste initially.

$$sN(s) = -\lambda N(s) + D(s)$$

$$(s+\lambda)N(s) = D(s)$$

$$N(s) = \frac{1}{s+\lambda} D(s) \equiv F(s)D(s)$$

$$\boxed{F(s) = \frac{1}{s+\lambda}}$$

$$\boxed{f(t) = e^{-\lambda t}}$$

$$n(t) = d(t) * f(t) = \int_0^t d(u) f(t-u) du$$

$$\boxed{n(t) = \int_0^t d(u) e^{-\lambda(t-u)} du}$$

Extra Bonus Solution. For $0 \le t \le 1$. **This Solution is not required as you were not asked to do this region. We include it to give you more. Then, we will give the regular solution asked for.**

$$n(t) = d(t) * f(t) = \int_0^t (1-u)e^{-\lambda(t-u)}du$$

$$n(t) = e^{-\lambda t}\int_0^t (1-u)e^{\lambda u}du$$

$$n(t) = e^{-\lambda t}\int_0^t e^{\lambda u}du - e^{-\lambda t}\int_0^t ue^{\lambda u}du$$

$$n(t) = e^{-\lambda t}\left[1 - \frac{d}{d\lambda}\right]\int_0^t e^{\lambda u}du$$

$$n(t) = e^{-\lambda t}\left[1 - \frac{d}{d\lambda}\right]\left[\frac{e^{\lambda u}}{\lambda}\right]_0^t$$

$$n(t) = e^{-\lambda t}\left[1 - \frac{d}{d\lambda}\right]\left[\frac{e^{\lambda t}-1}{\lambda}\right]$$

$$n(t) = e^{-\lambda t}\left[\frac{e^{\lambda t}-1}{\lambda}\right] - e^{-\lambda t}\left[(-\frac{1}{\lambda^2})(e^{\lambda t}-1) + (\frac{1}{\lambda})te^{\lambda t}\right]$$

$$n(t) = e^{-\lambda t}\left[\frac{e^{\lambda t}-1}{\lambda} + \frac{1}{\lambda^2}(e^{\lambda t}-1) - \frac{1}{\lambda}te^{\lambda t}\right]$$

Check: Does $n(0) = 0$? Yes, we are good there. See below for details.

$$n(0) = e^{-0}\left[\frac{e^{0}-1}{\lambda} + \frac{1}{\lambda^2}(e^{0}-1) - \frac{1}{\lambda}0 \cdot e^{0}\right] = 0$$

The Solution. For $t \geq 1$ **This Solution is what you were asked to do.**

$$n(t) = d(t) * f(t) = \int_0^t (1-u)e^{-\lambda(t-u)} du$$

$$n(t) = e^{-\lambda t} \int_0^1 (1-u)e^{\lambda u} du$$

$$n(t) = e^{-\lambda t} \int_0^1 e^{\lambda u} du - e^{-\lambda t} \int_0^1 u e^{\lambda u} du$$

$$n(t) = e^{-\lambda t} \left[1 - \frac{d}{d\lambda}\right] \int_0^1 e^{\lambda u} du$$

$$n(t) = e^{-\lambda t} \left[1 - \frac{d}{d\lambda}\right] \left[\frac{e^{\lambda u}}{\lambda}\right]_0^1$$

$$n(t) = e^{-\lambda t} \left[1 - \frac{d}{d\lambda}\right] \left[\frac{e^{\lambda} - 1}{\lambda}\right]$$

$$n(t) = e^{-\lambda t} \left[\frac{e^{\lambda} - 1}{\lambda}\right] - e^{-\lambda t} \left[(-\frac{1}{\lambda^2})(e^{\lambda} - 1) + (\frac{1}{\lambda})e^{\lambda}\right]$$

$$n(t) = e^{-\lambda t} \left[\frac{e^{\lambda} - 1}{\lambda} + \frac{1}{\lambda^2}(e^{\lambda} - 1) - \frac{1}{\lambda} e^{\lambda}\right]$$

$$n(t) = e^{-\lambda t} \left[-\frac{1}{\lambda} + \frac{1}{\lambda^2}(e^{\lambda} - 1)\right]$$

$$\boxed{n(t) = \frac{e^{-\lambda t}}{\lambda} \left[\frac{e^{\lambda} - 1}{\lambda} - 1\right]}$$

The following is another extra bonus for you.

Shortcut if you have the $0 \le t \le 1$ solution (extra stuff given above). Start with oue bonus solution from before.

$$n(t) = e^{-\lambda t}\left[\frac{e^{\lambda t}-1}{\lambda} + \frac{1}{\lambda^2}(e^{\lambda t}-1) - \frac{1}{\lambda} t e^{\lambda t}\right] \quad \text{for} \quad 0 \le t \le 1 .$$

$$n(1) = e^{-\lambda}\left[\frac{e^{\lambda}-1}{\lambda} + \frac{1}{\lambda^2}(e^{\lambda}-1) - \frac{1}{\lambda} e^{\lambda}\right]$$

$$n(1) = e^{-\lambda}\left[-\frac{1}{\lambda} + \frac{1}{\lambda^2}(e^{\lambda}-1)\right]$$

$$n(1) = \left[-\frac{e^{-\lambda}}{\lambda} + \frac{1}{\lambda^2}(1-e^{-\lambda})\right]$$

Now apply the time shifted decay to this since you are not adding any more dumping.

$$n(t) = \left[-\frac{e^{-\lambda}}{\lambda} + \frac{1}{\lambda^2}(1-e^{-\lambda})\right] e^{-\lambda(t-1)}$$

$$n(t) = e^{-\lambda t} e^{\lambda}\left[-\frac{e^{-\lambda}}{\lambda} + \frac{1}{\lambda^2}(1-e^{-\lambda})\right] = e^{-\lambda t}\left[-\frac{1}{\lambda} + \frac{e^{\lambda}}{\lambda^2}(1-e^{-\lambda})\right]$$

$$n(t) = e^{-\lambda t}\left[-\frac{1}{\lambda} + \frac{e^{\lambda}}{\lambda^2} - \frac{1}{\lambda^2})\right] = \frac{e^{-\lambda t}}{\lambda}\left[-1 + \frac{e^{\lambda}}{\lambda} - \frac{1}{\lambda})\right]$$

$$\boxed{n(t) = \frac{e^{-\lambda t}}{\lambda}\left[\frac{e^{\lambda}-1}{\lambda} - 1\right]}$$ This result is the same as we found before.