Theoretical Physics

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Chapter O Homework Solutions. Fourier Series

HW-O1. Orthogonality of the Cosine Functions. Show that

$$\int_{-\pi}^{+\pi} \cos(nx)\cos(mx)dx = \pi\delta_{nm}$$

You must do your integrals using the backward Euler formula as we did in the class notes for the sine case. Then use your above result to show that

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos(nx)dx \; .$$

Part A. When $n = m$

$$\int_{-\pi}^{+\pi} \cos^2(nx)dx = ?$$

$$\cos(nx) = \frac{e^{inx} + e^{-inx}}{2}$$

$$\cos^2(nx) = \frac{e^{2inx} + 2 + e^{-2inx}}{4}$$

Part A1. The first exponential integral in $\cos^2(nx)$

$$\int_{-\pi}^{+\pi} e^{2inx}dx = \left.\frac{e^{2inx}}{2in}\right|_{-\pi}^{+\pi} = \frac{1}{2in}\left[\cos(2nx) + i\sin(2nx)\right]\Big|_{-\pi}^{+\pi}$$

$$= \frac{1}{2in}\left[\cos(2n\pi) + i\sin(2n\pi)\right] - \frac{1}{2in}\left[\cos(-2n\pi) + i\sin(-2n\pi)\right]$$

$$= \frac{1}{2in}\left[\cos(2n\pi) + i\sin(2n\pi)\right] - \frac{1}{2in}\left[\cos(2n\pi) - i\sin(2n\pi)\right]$$

$$= \frac{1}{2in}\left[\cos(2n\pi) - \cos(2n\pi)\right] - \frac{1}{2in}\left[\sin(2n\pi) - i\sin(2n\pi)\right]$$

$$= \frac{1}{2in}[0] - \frac{1}{2in}[0 - 0] = 0$$

Part A2. The the other exponential integral in $\cos^2(nx)$

$$\int_{-\pi}^{+\pi} e^{-2inx}dx = \left.\frac{e^{-2inx}}{-2in}\right|_{-\pi}^{+\pi} = \frac{1}{2in}\left[\cos(-2nx) + i\sin(-2nx)\right]\Bigg|_{-\pi}^{+\pi}$$

$$\int_{-\pi}^{+\pi} e^{-2inx}dx = \left.\frac{e^{-2inx}}{-2in}\right|_{-\pi}^{+\pi} = \frac{1}{2in}\left[\cos(2nx) - i\sin(2nx)\right]\Bigg|_{-\pi}^{+\pi}$$

$$= \frac{1}{2in}\left[\cos(2n\pi) - i\sin(2n\pi)\right] - \frac{1}{2in}\left[\cos(-2n\pi) - i\sin(-2n\pi)\right]$$

$$= \frac{1}{2in}\left[\cos(2n\pi) - i\sin(2n\pi)\right] - \frac{1}{2in}\left[\cos(2n\pi) + i\sin(2n\pi)\right]$$

$$= \frac{1}{2in}\left[\cos(2n\pi) - \cos(2n\pi)\right] - \frac{1}{2in}\left[-\sin(2n\pi) + i\sin(2n\pi)\right]$$

$$= \frac{1}{2in}[0] - \frac{1}{2in}[-0 + 0] = 0$$

Part A3. The constant part in $\cos^2(nx)$ is all we need.

$$\int_{-\pi}^{+\pi} \cos^2(nx)dx = \int_{-\pi}^{+\pi} \frac{1}{2} dx = \frac{x}{2}\Big|_{-\pi}^{+\pi} = \frac{1}{2}[\pi - (-\pi)] = \pi$$

Part B. When $n \neq m$

$$\cos(nx)\cos(mx) = \left[\frac{e^{inx} + e^{-inx}}{2}\right]\left[\frac{e^{imx} + e^{-imx}}{2}\right]$$

Since $n \neq m$, all integrals have the form

$$\int_{-\pi}^{+\pi} e^{ipx} dx = \frac{e^{ipx}}{ip}\Big|_{-\pi}^{+\pi} = \frac{1}{ip}[\cos(px) + i\sin(px)]\Big|_{-\pi}^{+\pi} = 0$$

The functions are said to be orthogonal. The integral serves as the dot product.

Summary:

$$\int_{-\pi}^{+\pi} \cos(nx)\cos(mx)dx = \pi\delta_{nm}$$

Part B. Show $a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos(nx)dx$

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}[a_m \cos(mx) + b_m \sin(mx)]$$

$$\int_{-\pi}^{+\pi} f(x)\cos(nx)dx$$

$$= \frac{a_0}{2}\int_{-\pi}^{+\pi} \cos(nx)dx + \sum_{m=1}^{\infty} a_m \int_{-\pi}^{+\pi} \cos(nx)\cos(mx)dx$$

$$+\sum_{m=1}^{\infty} b_m \int_{-\pi}^{+\pi} \cos(nx)\sin(mx)dx$$

Part B1. The First Integral.

$$\frac{a_0}{2}\int_{-\pi}^{+\pi} \cos(nx)dx = \frac{a_0}{2}\frac{\sin(nx)}{n}\bigg|_{-\pi}^{+\pi} = 0$$

Part B2. The Second Integral.

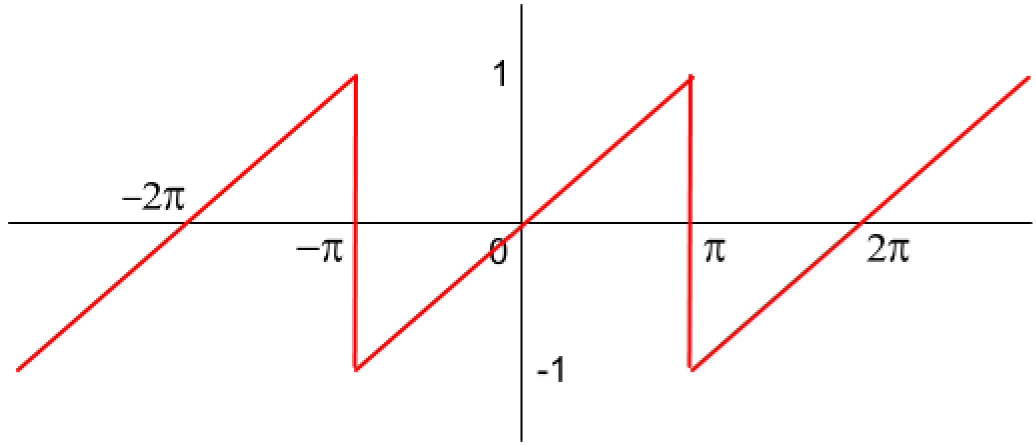
$$\sum_{m=1}^{\infty} a_m \int_{-\pi}^{+\pi} \cos(nx)\cos(mx)dx = \sum_{m=1}^{\infty} a_m \pi \delta_{nm} = \pi a_n$$

Part B3. The Third Integral. This is zero since our integrand is odd and we are integrating over a symmetric region.

Therefore

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos(nx)dx \; .$$

HW-O2. The Ramp or Sawtooth Waveform. Find the Fourier amplitudes for the ramp wave which has an amplitude of 1.



$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left[a_m \cos(mx) + b_m \sin(mx) \right]$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos(nx) dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin(nx) dx$$

Since our function is odd, we will only have b_n

$$b_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin(nx) dx$$

$$f(x) = \frac{x}{\pi}$$

$$b_n = \frac{1}{\pi^2} \int_{-\pi}^{+\pi} x \sin(nx) dx$$

$$b_n = \frac{2}{\pi^2}\int_0^{+\pi} x\sin(nx)dx$$

From the integral tables: $\int x\sin(ax)dx = -\frac{x\cos ax}{a} + \frac{\sin ax}{a^2} + C$

$$b_n = \frac{2}{\pi^2}\int_0^{+\pi} x\sin(nx)dx = \frac{2}{\pi^2}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$b_n = \frac{2}{\pi^2}\left[-\frac{\pi\cos n\pi}{n} + \frac{\sin n\pi}{n^2}\right] - \frac{2}{\pi^2}\left[-\frac{(0)\cos(0)}{n} + \frac{\sin(0)}{n^2}\right]$$

$$b_n = \frac{2}{\pi^2}\left[-\frac{\pi\cos n\pi}{n} + \frac{\sin n\pi}{n^2}\right]$$

$$b_n = \frac{2}{\pi^2}\left[-\frac{\pi\cos n\pi}{n}\right] = -\frac{2}{\pi}\left[-\frac{\cos n\pi}{n}\right]$$

For odd n: $b_n = -\frac{2}{\pi}\left[\frac{(-1)}{n}\right] = \frac{2}{\pi}\frac{1}{n}$

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For even n: $b_n = \frac{2}{\pi}\left[-\frac{(+1)}{n}\right] = -\frac{2}{\pi}\frac{1}{n}$

$$\boxed{f(x) = \frac{2}{\pi}\left[\sin x - \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) - \frac{1}{4}\sin(4x)...\right]}$$