Theoretical Physics
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Chapter L Homework. The Dirac Equation

HW-L1. Dirac Matrices. Show that $\alpha_2\alpha_3 + \alpha_3\alpha_2 = 0$ by explicitly multiplying out the 4 x 4 matrices. Then use the shortcut method with $\alpha_2 = \begin{bmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{bmatrix}$ and $\alpha_3 = \begin{bmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{bmatrix}$ to show the same result.

$$\alpha_2\alpha_3 + \alpha_2\alpha_3 =$$

$$\begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{bmatrix} + \begin{bmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 0$$

Shortcut Method.

$$\alpha_2\alpha_3 + \alpha_2\alpha_3 = \begin{bmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{bmatrix} + \begin{bmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_y\sigma_z & 0 \\ 0 & \sigma_y\sigma_z \end{bmatrix} + \begin{bmatrix} \sigma_z\sigma_y & 0 \\ 0 & \sigma_z\sigma_y \end{bmatrix} = \begin{bmatrix} \sigma_y\sigma_z + \sigma_z\sigma_y & 0 \\ 0 & \sigma_y\sigma_z + \sigma_z\sigma_y \end{bmatrix}$$

$$\alpha_2\alpha_3 + \alpha_2\alpha_3 = \begin{bmatrix} \sigma_y\sigma_z + \sigma_z\sigma_y & 0 \\ 0 & \sigma_y\sigma_z + \sigma_z\sigma_y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

We used $\sigma_y\sigma_z + \sigma_z\sigma_y = 0$ in the last step.

HW-L2. Pauli Matrices.

Show $(\hat{n}\cdot\vec{\sigma})^{2k} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $(\hat{n}\cdot\vec{\sigma})^{2k+1} = \hat{n}\cdot\vec{\sigma}$ for $k = 0,1,2,\ldots$.

$$\hat{n}\cdot\vec{\sigma} = \sin\theta\cos\phi \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \sin\theta\sin\phi \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \cos\theta \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \sin\theta \begin{bmatrix} 0 & \cos\phi \\ \cos\phi & 0 \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & -i\sin\phi \\ i\sin\phi & 0 \end{bmatrix} + \cos\theta \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\hat{n}\cdot\vec{\sigma} = \sin\theta \begin{bmatrix} 0 & \cos\phi - i\sin\phi \\ \cos\phi + i\sin\phi & 0 \end{bmatrix} + \cos\theta \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\hat{n}\cdot\vec{\sigma} = \begin{bmatrix} \cos\theta & \sin\theta e^{-i\theta} \\ \sin\theta e^{i\theta} & -\cos\theta \end{bmatrix}$$

$$(\hat{n}\cdot\vec{\sigma})^2 = \begin{bmatrix} \cos\theta & \sin\theta e^{-i\theta} \\ \sin\theta e^{i\theta} & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta e^{-i\theta} \\ \sin\theta e^{i\theta} & -\cos\theta \end{bmatrix}$$

$$(\hat{n}\cdot\vec{\sigma})^2 = \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$(\hat{n}\cdot\vec{\sigma})^3 = (\hat{n}\cdot\vec{\sigma})(\hat{n}\cdot\vec{\sigma})^2 = (\hat{n}\cdot\vec{\sigma})I = (\hat{n}\cdot\vec{\sigma})$$

Continuing in this fashion,

$(\hat{n}\cdot\vec{\sigma})^{2k} = I$ and $(\hat{n}\cdot\vec{\sigma})^{2k+1} = \hat{n}\cdot\vec{\sigma}$ for $k = 0,1,2,\ldots$

HW-L3. Generating SU(2) Matrices. Show that

$$e^{i\alpha\hat{n}\cdot\vec{\sigma}} = I\cos\alpha + i\hat{n}\cdot\vec{\sigma}\sin\alpha \,.$$

As directed from HW-K1, use $e^{A} = I + A + \frac{1}{2!}A^{2} + \frac{1}{3!}A^{3} + \ldots$ for $e^{i\alpha\hat{n}\cdot\vec{\sigma}}$.

$$e^{i\alpha\hat{n}\cdot\vec{\sigma}} = I + i\alpha\hat{n}\cdot\vec{\sigma} + \frac{1}{2!}(i\alpha\hat{n}\cdot\vec{\sigma})^{2} + \frac{1}{3!}(i\alpha\hat{n}\cdot\vec{\sigma})^{3} + \frac{1}{4!}(i\alpha\hat{n}\cdot\vec{\sigma})^{4}\ldots$$

$$e^{i\alpha\hat{n}\cdot\vec{\sigma}} = I + i\alpha\hat{n}\cdot\vec{\sigma} - \frac{\alpha^{2}}{2!}(\hat{n}\cdot\vec{\sigma})^{2} - i\frac{\alpha^{3}}{3!}(\hat{n}\cdot\vec{\sigma})^{3} + \frac{\alpha^{4}}{4!}(\hat{n}\cdot\vec{\sigma})^{4}\ldots$$

From the last problem: $(\hat{n}\cdot\vec{\sigma})^{2k} = I$ and $(\hat{n}\cdot\vec{\sigma})^{2k+1} = \hat{n}\cdot\vec{\sigma}$ for $k = 0, 1, 2, \ldots$

Therefore, $e^{i\alpha\hat{n}\cdot\vec{\sigma}} = I + i\alpha\hat{n}\cdot\vec{\sigma} - \frac{\alpha^{2}}{2!}I - i\frac{\alpha^{3}}{3!}\hat{n}\cdot\vec{\sigma} + \frac{\alpha^{4}}{4!}I\ldots$.

$$e^{i\alpha\hat{n}\cdot\vec{\sigma}} = I\left[1 - \frac{\alpha^{2}}{2!} + \frac{\alpha^{4}}{4!}\ldots\right] + i\left[\alpha - \frac{\alpha^{3}}{3!}\ldots\right]\hat{n}\cdot\vec{\sigma}$$

Now we use the expansions for cosine and sine.

$$\cos\alpha = 1 - \frac{\alpha^{2}}{2!} + \frac{\alpha^{4}}{4!} - \ldots$$

$$\sin\alpha = \alpha - \frac{\alpha^{3}}{3!} + \frac{\alpha^{5}}{5!} - \ldots$$

$$e^{i\alpha\hat{n}\cdot\vec{\sigma}} = I\cos\alpha + i\hat{n}\cdot\vec{\sigma}\sin\alpha$$

HW-L4. Unitary Matrices. Show that $U = e^{i\alpha \hat{n}\cdot\vec{\sigma}} = I\cos\alpha + i\hat{n}\cdot\vec{\sigma}\sin\alpha$ is unitary.

From the given $A = \begin{bmatrix} a_r + ia_i & b_r + ib_i \\ -b_r + ib_i & a_r - ia_i \end{bmatrix}$ is unitary if the a and b factors are real.

We know $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\hat{n}\cdot\vec{\sigma} = \begin{bmatrix} \cos\theta & \sin\theta e^{-i\theta} \\ \sin\theta e^{i\theta} & -\cos\theta \end{bmatrix}$ from HW-L2.

Therefore, $U = e^{i\alpha \hat{n}\cdot\vec{\sigma}} = I\cos\alpha + i\hat{n}\cdot\vec{\sigma}\sin\alpha$ becomes

$$U = e^{i\alpha \hat{n}\cdot\vec{\sigma}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\cos\alpha + i\sin\alpha\begin{bmatrix} \cos\theta & \sin\theta e^{-i\theta} \\ \sin\theta e^{i\theta} & -\cos\theta \end{bmatrix}$$

$$U = e^{i\alpha \hat{n}\cdot\vec{\sigma}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\cos\alpha + i\sin\alpha\begin{bmatrix} \cos\theta & \sin\theta e^{-i\theta} \\ \sin\theta e^{i\theta} & -\cos\theta \end{bmatrix}$$

$$U = \begin{bmatrix} \cos\alpha + i\sin\alpha\cos\theta & i\sin\alpha\sin\theta e^{-i\theta} \\ i\sin\alpha\sin\theta e^{i\theta} & \cos\alpha - i\sin\alpha\cos\theta \end{bmatrix}$$

$$U = \begin{bmatrix} \cos\alpha + i\sin\alpha\cos\theta & i\sin\alpha\sin\theta(\cos\theta - i\sin\theta) \\ i\sin\alpha\sin\theta(\cos\theta + i\sin\theta) & \cos\alpha - i\sin\alpha\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha + i\sin\alpha\cos\theta & \sin\alpha\sin^2\theta - i\sin\alpha\sin\theta\cos\theta \\ -\sin\alpha\sin^2\theta + i\sin\alpha\sin\theta\cos\theta & \cos\alpha - i\sin\alpha\cos\theta \end{bmatrix}$$

Comparing with $A = \begin{bmatrix} a_r + ia_i & b_r + ib_i \\ -b_r + ib_i & a_r - ia_i \end{bmatrix}$ => $a_r = \cos\alpha$, $a_i = \sin\alpha\cos\theta$,
$b_r = \sin\alpha\sin^2\theta$, and $b_i = \sin\alpha\sin\theta$. Since these are real, our matrix is unitary.

HW-L5. Special Unitary Matrices. Show that $U = e^{i\alpha \hat{n}\cdot\vec{\sigma}} = I\cos\alpha + i\hat{n}\cdot\vec{\sigma}\sin\alpha$ is also a special matrix, i.e., the determinant $\det U = 1$.

We can start with our expression from the last problem.

$$U = \begin{bmatrix} \cos\alpha + i\sin\alpha\cos\theta & i\sin\alpha\sin\theta e^{-i\theta} \\ i\sin\alpha\sin\theta e^{i\theta} & \cos\alpha - i\sin\alpha\cos\theta \end{bmatrix}$$

For the special condition, we calculate the determinant.

$$\det U = (\cos\alpha + i\sin\alpha\cos\theta)(\cos\alpha - i\sin\alpha\cos\theta)$$

$$-(i\sin\alpha\sin\theta e^{i\theta})(i\sin\alpha\sin\theta e^{-i\theta})$$

$$\det U = \cos^2\alpha + \sin^2\alpha\cos^2\theta + \sin^2\alpha\sin^2\theta$$

$$\det U = \cos^2\alpha + \sin^2\alpha(\cos^2\theta + \sin^2\theta)$$

$$\det U = \cos^2\alpha + \sin^2\alpha$$

$$\det U = 1$$