Theoretical Physics
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Chapter K Homework. The Pauli Equation

HW-K1. Matrix in an Exponential. We define an matrix in an exponential as follows:

$$e^{A} = I + A + \frac{1}{2!}A^{2} + \frac{1}{3!}A^{3} + \ldots$$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and

$A^{2} = AA$, $A^{3} = AAA$, and so on (matrix multiplication).

Give the simplest result for e^{A} for $A = \theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Hint: You are obviously not expected to calculate an infinite amount of matrix multiplications as you would never finish your program at UNCA. Instead, you will see patterns. To appreciate these patterns, first play around with various multiplications of $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ with itself over and over again. Then investigate the same with A.

HW-K2. Wave Function with Spin. First note the definition for multiplying a row matrix with a column matrix:

$$\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = ac + bd$$. You will need this.

Dirac defined a "bra" and "ket" from breaking apart the word "bracket." A bracket can be visualized as $<>$.

The "ket" is $\left|\psi\right\rangle \equiv \begin{bmatrix} f(x) \\ g(x) \end{bmatrix}$ and the "bra" is $\left\langle\psi\right| \equiv \begin{bmatrix} f^*(x) & g^*(x) \end{bmatrix}$.

For a matrix $A = \begin{bmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{bmatrix}$,

$$\langle \psi | A | \psi \rangle \equiv \int_{-\infty}^{\infty} [\, f^* \quad g^*] \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} dx$$

One works out the matrix multiplications and then integrates over all x.

Consider the wave function ψ :

$$\psi = 0 \text{ for } -\infty \leq x < 0$$

$$\psi = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-x/2} \text{ for } 0 \leq x \leq \infty \text{ .}$$

and the Hermitian matrix operator:

$$A = \begin{bmatrix} x & i \\ -i & x \end{bmatrix}.$$

Calculate

$$\langle \psi | A | \psi \rangle.$$