**Theoretical Physics**
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**Chapter J Homework. Spinors**

**HW-J1. Matrix Properties.** You are given $\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$ and the Pauli matrices

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \text{ and } \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Show that the matrix $M = \vec{\sigma} \cdot \vec{A} = \begin{bmatrix} A_z & A_x - iA_y \\ A_x + iA_y & -A_z \end{bmatrix}$. Then calculate each of the following six items:

$$Tr(M), \quad M^T, \quad M^*, \quad M^\dagger, \quad \det(M), \quad M^{-1}.$$

**Solution**

(a) $M = \vec{\sigma} \cdot \vec{A} = \sigma_x A_x + \sigma_y A_y + \sigma_z A_z$

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A_x + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} A_y + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} A_z$$

$$M = \begin{bmatrix} 0 & A_x \\ A_x & 0 \end{bmatrix} + \begin{bmatrix} 0 & -A_y i \\ A_y i & 0 \end{bmatrix} + \begin{bmatrix} A_z & 0 \\ 0 & -A_z \end{bmatrix}$$

$$M = \begin{bmatrix} A_z & A_x - iA_y \\ A_x + iA_y & -A_z \end{bmatrix}$$

(b) $Tr(M) = A_z + (-A_z) = 0$

(c) $M^T = \begin{bmatrix} A_z & A_x + iA_y \\ A_x - iA_y & -A_z \end{bmatrix}$

$$M = \begin{bmatrix} A_z & A_x - iA_y \\ A_x + iA_y & -A_z \end{bmatrix}$$

(d) $M^* = \begin{bmatrix} A_z & A_x + iA_y \\ A_x - iA_y & -A_z \end{bmatrix}$

(e) $M^\dagger = \begin{bmatrix} A_z & A_x - iA_y \\ A_x + iA_y & -A_z \end{bmatrix} = M$

(f) $\det(M) = |M| = -A_z^2 - (A_x + iA_y)(A_x - iA_y) = -A_z^2 - A_x^2 - A_y^2 = -A^2$

(g) For the inverse, use $A^{-1} = \dfrac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$M^{-1} = \frac{1}{-A^2}\begin{bmatrix} -A_z & -A_x + iA_y \\ -A_x - iA_y & A_z \end{bmatrix} = \frac{1}{A^2}\begin{bmatrix} A_z & A_x - iA_y \\ A_x + iA_y & -A_z \end{bmatrix}$$

(h) Check:

$$MM^{-1} = \frac{1}{A^2}\begin{bmatrix} A_z & A_x - iA_y \\ A_x + iA_y & -A_z \end{bmatrix}\begin{bmatrix} A_z & A_x - iA_y \\ A_x + iA_y & -A_z \end{bmatrix}$$

$$MM^{-1} = \frac{1}{A^2}\begin{bmatrix} A_z^2 + A_x^2 + A_y^2 & 0 \\ 0 & A_x^2 + A_y^2 + A_z^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

**HW-J2. Pauli Matrices Identity.** Show $\vec{\sigma}\cdot\vec{A}\,\vec{\sigma}\cdot\vec{B}=\vec{A}\cdot\vec{B}\,I+i\vec{\sigma}\cdot(\vec{A}\times\vec{B})$.

$$\vec{\sigma}\cdot\vec{A}\,\vec{\sigma}\cdot\vec{B}=(\sigma_x A_x+\sigma_y A_y+\sigma_z A_z)(\sigma_x B_x+\sigma_y B_y+\sigma_z B_z)$$

$$\vec{\sigma}\cdot\vec{A}\,\vec{\sigma}\cdot\vec{B}=\sigma_x\sigma_x A_x B_x+\sigma_x\sigma_y A_x B_y+\sigma_x\sigma_z A_x B_z$$

$$+\sigma_y\sigma_x A_y B_x+\sigma_y\sigma_y A_y B_y+\sigma_y\sigma_z A_y B_z$$

$$+\sigma_z\sigma_x A_z B_x+\sigma_z\sigma_y A_z B_y+\sigma_z\sigma_z A_z B_z$$

$$=\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}A_x B_x+\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0&-i\\i&0\end{bmatrix}A_x B_y+\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}A_x B_z$$

$$+\begin{bmatrix}0&-i\\i&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}A_y B_x+\begin{bmatrix}0&-i\\i&0\end{bmatrix}\begin{bmatrix}0&-i\\i&0\end{bmatrix}A_y B_y+\begin{bmatrix}0&-i\\i&0\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}A_y B_z$$

$$+\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}A_z B_x+\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}0&-i\\i&0\end{bmatrix}A_z B_y+\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}A_z B_z$$

$$=\begin{bmatrix}1&0\\0&1\end{bmatrix}A_x B_x+\begin{bmatrix}i&0\\0&-i\end{bmatrix}A_x B_y+\begin{bmatrix}0&-1\\1&0\end{bmatrix}A_x B_z$$

$$+\begin{bmatrix}-i&0\\0&i\end{bmatrix}A_y B_x+\begin{bmatrix}1&0\\0&1\end{bmatrix}A_y B_y+\begin{bmatrix}0&i\\i&0\end{bmatrix}A_y B_z$$

$$+\begin{bmatrix}0&1\\-1&0\end{bmatrix}A_z B_x+\begin{bmatrix}0&-i\\-i&0\end{bmatrix}A_z B_y+\begin{bmatrix}1&0\\0&1\end{bmatrix}A_z B_z$$

$$=IA_x B_x+i\sigma_z A_x B_y+i\sigma_y A_x B_z$$

$$-i\sigma_z A_y B_x+IA_y B_y+i\sigma_x A_y B_z$$

$$+i\sigma_y A_z B_x-i\sigma_x A_z B_y+I\,A_z B_z$$

$$=I(A_x B_x+A_y B_y+A_z B_z)+i\sigma_x(A_y B_z-A_z B_y)$$

$$=i\sigma_y(A_x B_z-A_z B_x)+i\sigma_z(A_x B_y-A_y B_x)$$

$$=I\,\vec{A}\cdot\vec{B}+i\sigma_x(\vec{A}\times\vec{B})_x+i\sigma_y(\vec{A}\times\vec{B})_y+i\sigma_z(\vec{A}\times\vec{B})_z$$

$$\vec{\sigma}\cdot\vec{A}\,\vec{\sigma}\cdot\vec{B}=\vec{A}\cdot\vec{B}\,I+i\vec{\sigma}\cdot(\vec{A}\times\vec{B})$$

## Alternative Advanced Grad-School Method

Prove $\vec{\sigma}\cdot\vec{A}\,\vec{\sigma}\cdot\vec{B}=\vec{A}\cdot\vec{B}\,I+i\vec{\sigma}\cdot(\vec{A}\times\vec{B})$.

$$\vec{\sigma}\cdot\vec{A}\,\vec{\sigma}\cdot\vec{B}=\sum_{i=1}^{3}\sigma_i A_i\sum_{j=1}^{3}\sigma_j A_j=\sigma_i A_i\sigma_j A_j$$

Einstein summation convention: You sum when an index appears twice.

$$\sigma_i\sigma_j=\delta_{ij}I+i\varepsilon_{ijk}\sigma_k$$

$$\vec{\sigma}\cdot\vec{A}\,\vec{\sigma}\cdot\vec{B}=A_iA_j\sigma_i\sigma_j=A_iA_j(\delta_{ij}I+i\varepsilon_{ijk}\sigma_k)$$

$$\vec{\sigma}\cdot\vec{A}\,\vec{\sigma}\cdot\vec{B}=A_iA_j\delta_{ij}I+i\varepsilon_{ijk}A_iA_j\sigma_k$$

$$\vec{\sigma}\cdot\vec{A}\,\vec{\sigma}\cdot\vec{B}=\vec{A}\cdot\vec{B}\,I+i(\vec{A}\times\vec{B})\cdot\vec{\sigma}$$

$$\vec{\sigma}\cdot\vec{A}\,\vec{\sigma}\cdot\vec{B}=\vec{A}\cdot\vec{B}\,I+i\vec{\sigma}\cdot(\vec{A}\times\vec{B})$$

REALLY ALL YOU NEED IS THE FOLLOWING ONE LINER:

$$\vec{\sigma}\cdot\vec{A}\,\vec{\sigma}\cdot\vec{B}=A_iA_j\sigma_i\sigma_j=A_iA_j(\delta_{ij}I+i\varepsilon_{ijk}\sigma_k)=\vec{A}\cdot\vec{B}\,I+i\vec{\sigma}\cdot(\vec{A}\times\vec{B})$$

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**HW-J3.** Find the eigenvalues and normalized eigenvectors for $\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$.

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \lambda \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} -\lambda & -i \\ i & -\lambda \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Set the determinant to zero.

$$\lambda^2 - (i)(-i) = 0$$

$$\lambda^2 + i^2 = 0$$

$$\lambda^2 = 1 \quad \text{and} \quad \lambda = \pm 1.$$

Two eigenvalues: $\lambda_1 = 1$ and $\lambda_2 = 1$.

For the first eigenvalue +1,

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \lambda_1 \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

$-ic_2 = c_1$, leading to $c_2 = ic_1$ and the normalized eigenvector $u = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}$.

For the second eigenvalue -1,

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \lambda_2 \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = -\begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

$-ic_2 = -c_1$, giving $c_2 = -ic_1$ and the normalized eigenvector $v = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}$.