Theoretical Physics
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Chapter H Homework. Statistical Mechanics

HW-H1. Two Quantum States Part I.

a) The Sketch. Sketch a dotted horizontal line to represent a zero reference energy point. Then draw a short solid horizontal line below this reference and label this line with energy $\varepsilon_1 = -\varepsilon$. Then draw a short solid horizontal line above your reference and label this line with energy $\varepsilon_2 = +\varepsilon$.

b) The Partition Function. At temperature T give the partition function for this system.

c) Occupation Numbers. Then give the most probable occupation numbers n_1 and n_2 for each of these states if the total number of particles is some large value $N = n_1 + n_2$.

d) Energy. Show that the average energy $\overline{E} = \dfrac{n_1\varepsilon_1 + n_2\varepsilon_2}{n_1 + n_2}$ can be expressed as hyperbolic function as follows: $\overline{E} = -\varepsilon \tanh \dfrac{\varepsilon}{kT}$. Then give a sketch of the average energy as a function of temperature.

HW-H2. Two Quantum States Part II. This problem is a continuation of HW-H1.

a) What are the occupation numbers and average energy in the limit as the temperature approaches absolute zero, i.e., $T \to 0$.

b) What are the occupation numbers and average energy in the limit as the temperature approaches infinity, i.e., $T \to \infty$.

c) Compare your results for (e) and (f) in terms of entropy, i.e., the disorder of the system, using words. Why do you expect your answers for (e) and (f) to come out the way they do in terms of entropy?

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HW-H3. Infinite Quantum States. Consider Max Planck's discrete energy levels given by $E_n = nhf$. Give the partition function as a summation from zero to infinity for these states.

This section need not be included in your homework write-up.

To evaluate your partition function, note that your sum is of the form

$$Z = 1 + e^{-x} + e^{-2x} + e^{-3x} + \dots, \text{ where } x > 0 .$$

Let $r = e^{-x}$. Then

$$Z = 1 + r + r^2 + r^3 + \dots \quad \text{and} \quad rZ = r + r^2 + r^3 + \dots .$$

Each series has an infinite number of terms, each getting smaller and smaller since $r = e^{-x} < 1$. Subtracting your two infinite sums,

$$Z - rZ = 1, \quad Z(1-r) = 1, \quad \text{and} \quad Z = \frac{1}{1-r} .$$

If you have never seen this, it is highly recommend that you derive the result for a finite sum of n terms: $S_n = a + ar + ar^2 + \dots + ar^{n-1} = a\left[\frac{1-r^n}{1-r}\right]$.

Use the following derivative trick in order to calculate your average energy.

$$\sum_n ne^{-nx} = -\frac{d}{dx}\sum_n e^{-nx} = -\frac{d}{dx}\left[\frac{1}{1-e^{-x}}\right]$$

Your final answer will be the prize: $\overline{E} = \dfrac{hf}{e^{\frac{hf}{kT}} - 1}$.

You get the classical result by making "h" small, just as you get the classical formulas from relativity by making "c" large. What is the limit for the energy when h goes to zero if you do a Taylor's expansion using $e^z \approx 1 + z$? The equipartition theorem in classical thermodynamics says you get $kT/2$ for each degree of freedom, i.e., each way the system can absorb energy. Remember the ideal gas with 3kT/2? Here, an oscillator has one degree for kinetic energy along the direction of the "spring" and one degree for its potential energy. So you should get kT for your classical average energy