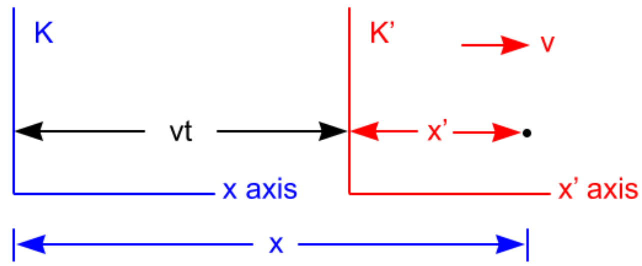
Theoretical Physics
Prof. Ruiz, UNC Asheville, doctorphys on YouTube
Chapter C Notes. Relativity

C1. Galilean Transformation



We consider inertial frames in this chapter. An inertial frame is one either at rest or moving in a straight line with constant speed. At the right are two such frames. The K frame can be considered at rest and the K' frame moving at speed v along the x-axis relative to the K frame.

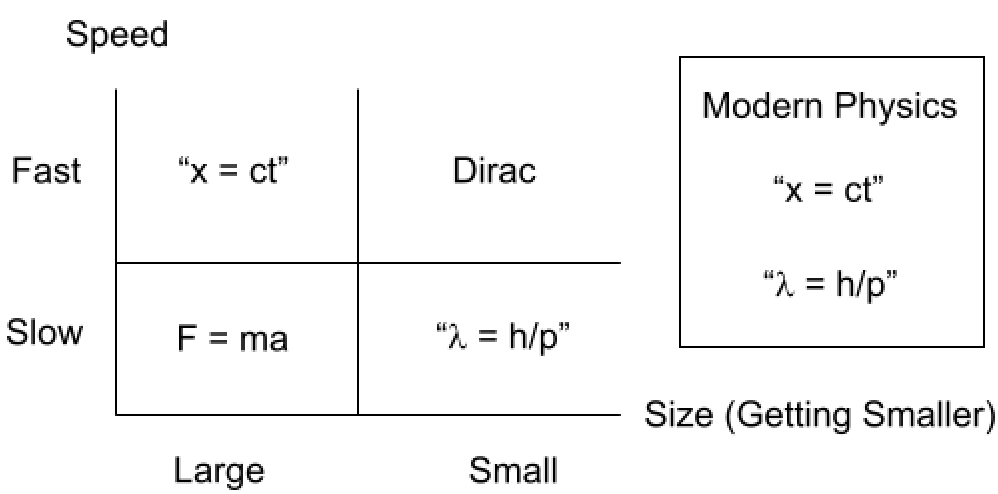
We can introduce the time coordinate for each frame, t and t', respectively for the K and K' frame. We synchronize the clocks so that t = t' = 0 when the origins overlap.

The common-sense classical relationship between the coordinates (x,t) in the K frame can readily be round since for any specific point x' we have x = vt + x'. We obtain

$$x' = x - vt$$

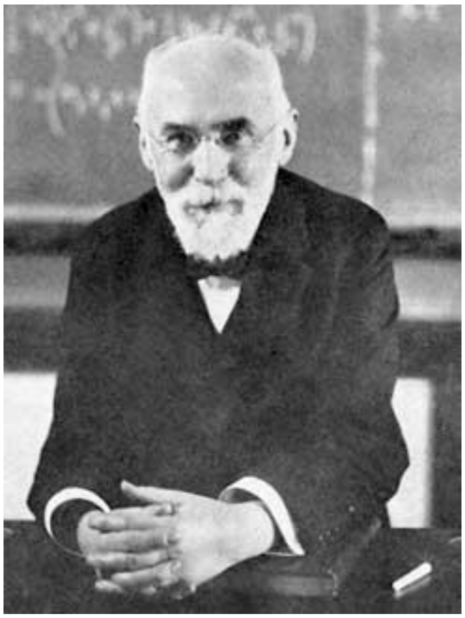
$$t' = t \, .$$

The second equation means absolute time. The two clocks run at the same rate. This is known as the Galilean transformation. However, this transformation is valid only for small speeds v. When the relative speed between the frames is great, i.e., appreciable compared to the speed of light, it is the Lorentz transformation that is valid.



C2. The Lorentz Transformation

This chapter might appear long by pages, but much of this chapter is a review of material from intro physics courses and modern physics. The Michelson-Morley experiment (1887) gave strange results that are ultimately understood with the speed of light as a constant in different frames of reference. Lorentz came up with the space transformation equation in 1895 relating x and x' in inertial frames to make things come out right. Then in 1899 he added the time transformation. But in those early days there was no fundamental theory as to why this should be the case.



Hendrik Antoon Lorentz (1853-1928)
Courtesy School of Mathematics and Statistics
University of St. Andrews, Scotland

The Lorentz transformation between the frames K and K':

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{and} \quad t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} .$$

The Dutch physicist Lorentz by the way shared the Nobel Prize in Physics in 1902 with Zeeman for the discovery and explanation of the Zeeman effect.

C3. Special Relativity

Albert Einstein (1879-1955)
Courtesy School of Mathematics and Statistics
University of St. Andrews, Scotland

Albert Einstein put forth the Theory of Special Relativity in 1905. The postulates are:

1. First Postulate. The Laws of Physics are the same in all inertial frames.

2. Second Postulate: The speed of light (in vacuum) is the same in all inertial frames.

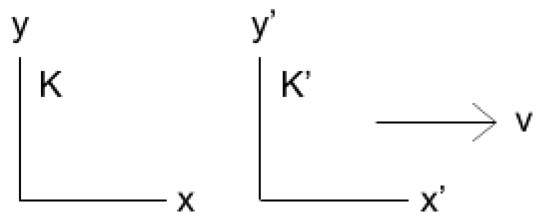
From the second postulate we can derive the Lorentz transformation. Einstein also published in 1905 his famous paper on the Photoelectric Effect, which won him the 1921 Nobel Prize in Physics. And this is not all he published that year.

We will derive the Lorentz transformation in a Feynmanesque manner, i.e., in a very elegant way.

Consider a light beam emitted at the origin when the clocks are synchronized at t = t'. For the two frames we have

$$x = ct \text{ and } x' = ct',$$

where the speed of light is taken to be the same according to Einstein's Second Postulate.



To allow for light traveling in the positive x and negative x directions we square both sides of each equation, arriving at

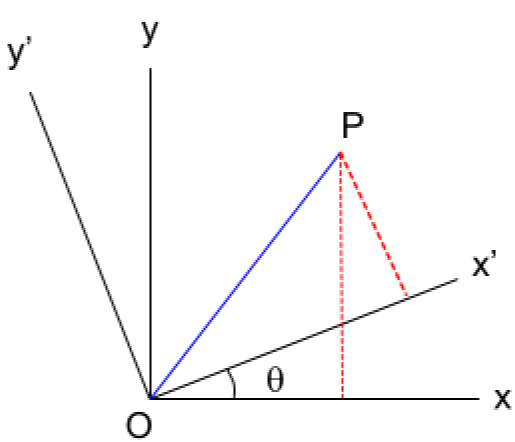
$$x^2 = c^2t^2 \text{ and } (x')^2 = c^2(t')^2.$$

We can then state the following elegant result.

$$x^2 - c^2t^2 = (x')^2 - c^2(t')^2$$

This arrangement is the same in each frame. We say that the quantity $x^2 - c^2t^2$ is invariant. Einstein was always looking for that which does not change from frame to frame such as the speed of light c and relations like this one.

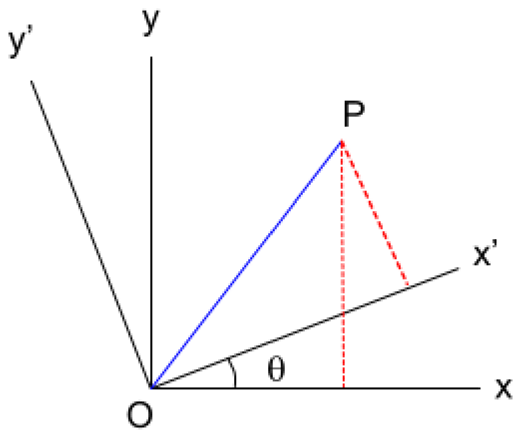
What is invariant in our rotation scheme below? The length of the OP line. Coordinates may vary but each frame agrees on the distance between O and P.



The mathematician Hermann Minkowski introduced the trick of letting $y = ict$. Then, the invariant is

$$x^2 + y^2 = x^2 - c^2t^2.$$

WARNING: This y is NOT our spatial y dimension in the reference frame above. The y = ict is related to our time variable. We still have our regular y in our room.



Our coordinate transformation

$$x' = x\cos\theta + y\sin\theta$$

$$y' = -x\sin\theta + y\cos\theta$$

with the Minkowski trick: $y = ict$ and $y' = ict'$, gives us the Lorentz transformation - well, almost. We still have to deal with the angle.

$$x' = x\cos\theta + ict\sin\theta$$

$$ict' = -x\sin\theta + ict\cos\theta$$

We have an analogy here. The rotations with angles are analogous to our reference frames with various speeds v. All we have to do is determine how the angle theta relates to the speed. We do this by looking at a point that stays put in the K' frame so that $\Delta x' = 0$.

Then, the observed change in that position measured by the K frame is $v = \frac{\Delta x}{\Delta t}$ as the K frame watches the K' frame moving on. We can write

$$\Delta x' = \Delta x\cos\theta + ic\Delta t\sin\theta = 0$$

$$\Delta x\cos\theta = -ic\Delta t\sin\theta$$

$$\frac{\Delta x}{\Delta t} = -ic\frac{\sin\theta}{\cos\theta}$$

$$v = -ic\tan\theta$$

$$\tan\theta = -\frac{v}{ic} = i\frac{v}{c}$$

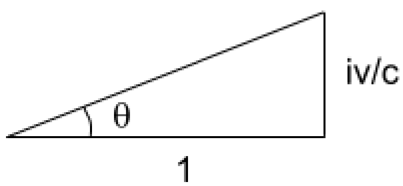
Now this would freak out most folks. The tangent is a slope. You can't have an imaginary slope. Thus we meet a characteristic of Feynman magic with math. Such an imaginary slope would not phase Feynman in the least and he would continue on as follows. In summary, we have so far

$$x' = x\cos\theta + ict\sin\theta$$

$$ict' = -x\sin\theta + ict\cos\theta$$

$$\tan\theta = i\frac{v}{c}$$

Proceeding as if nothing is strange, we set up our right triangle so that the tangent of the angle is correct. We then determine the hypotenuse with Pythagorean's Theorem, not phased in the least with the imaginary number.



$$hypotenuse = \sqrt{1^2 + \left[i\frac{v}{c}\right]^2} = \sqrt{1-\frac{v^2}{c^2}}$$

This is amazing. Filling in for the trig we have

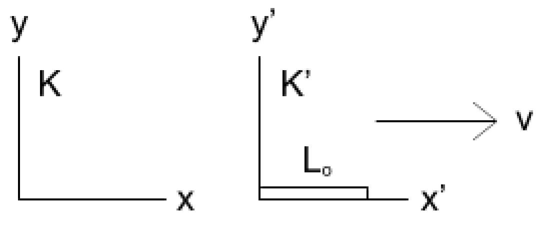
$$x' = x\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} + ict\frac{i\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}}$$, which gives $$x' = \frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}}$$;

$$ict' = -x\frac{i\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}} + ict\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$, which gives $$t' = \frac{t-x\frac{v}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$.

I first saw this cool derivation in L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamum Press, Oxford, 1962).

C4. Space and Time

1) Lorentz Contraction. Place a rod of length L_o in the K' frame. What is the length as measured from the K frame?



We start with

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$ and take deltas.

$$\Delta x' = \frac{\Delta x - v\Delta t}{\sqrt{1 - \frac{v^2}{c^2}}}$$

We have $\Delta x' = L_o$ and $\Delta x = L$, the second being the measurement as the rod zips by us. When we make our measurement we want to "nail" the ends at the same instant. Therefore, our measurement takes place such that $\Delta t = 0$. We then have

$$L_o = \frac{L}{\sqrt{1 - \frac{v^2}{c^2}}}$$ and $$L = L_o\sqrt{1 - \frac{v^2}{c^2}}\ .$$

We measure the moving rod to be contracted. Note that the proper length of the rod is L_o, the length as measured in the frame in which the rod is at rest.

2) Time Dilation. Place a clock in the K' frame. We need to keep the clock still in that frame. What is the time as measured from the K frame? Since the clock is at rest in the K' frame, $\Delta x' = 0$. The proper time T_o is time kept by this stationary clock in its own K' frame. Therefore, $T_o = \Delta t'$. The time $T = \Delta t$ is the time measured in the K frame as the clock zips by. We need to use the equation of the Lorentz transformation that has x', t', and t, so we can find a $\Delta x'$, $\Delta t'$, and Δt. We do not see one since we have

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$ and $$t' = \frac{t - x\frac{v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}\ .$$

So we write the inverse transformation, i.e., how to get the K coordinates from the K' coordinates. This is easy. You just let v go to minus v. You put yourself in the K' frame and watch the K frame go the other way.

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{and} \quad t' = \frac{t - x\frac{v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{then become}$$

$$x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{and} \quad t = \frac{t' + x'\frac{v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} .$$

Now we can obtain an equation with $\Delta x'$, $\Delta t'$, and Δt. We use the last one and take deltas.

$$\Delta t = \frac{\Delta t' + \Delta x'\frac{v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} , \quad \text{which becomes} \quad T = \frac{T_0 + 0 \cdot \frac{v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} .$$

The result is time dilation, a stretching of time:

$$T = \frac{T_0}{\sqrt{1 - \frac{v^2}{c^2}}} .$$

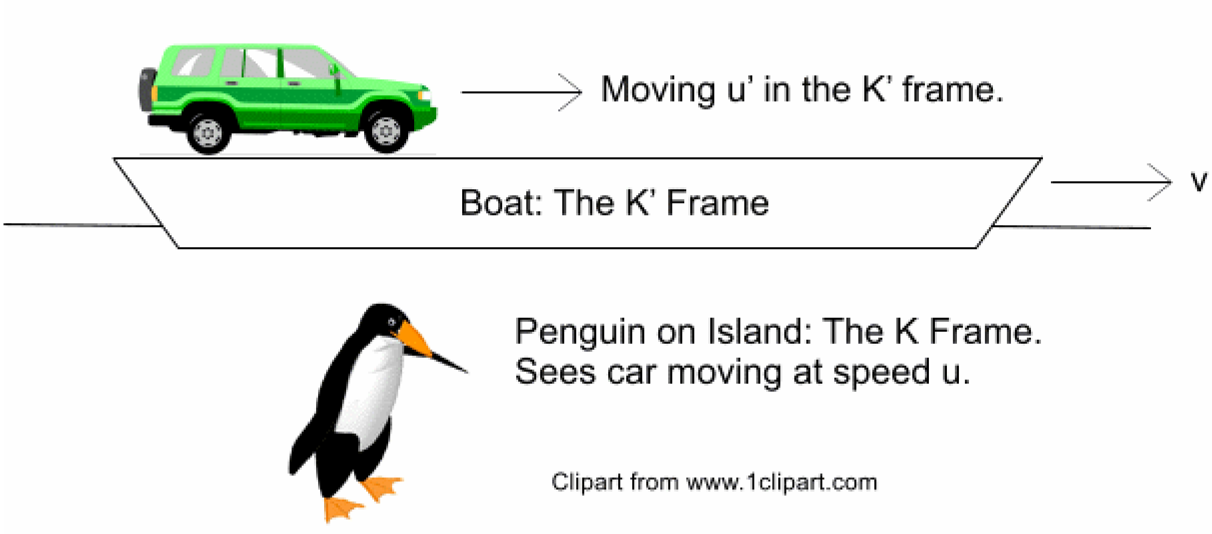
Summary:

Lorentz Contraction of the moving rod: $L = L_o\sqrt{1 - \frac{v^2}{c^2}}$ as we measure it go by.

Time Dilation of the moving clock: $T = \frac{T_0}{\sqrt{1 - \frac{v^2}{c^2}}}$ as we watch is go by. Our time is longer.

C5. Velocity Addition

A car is driving such that u' registers on its speedometer, but it is traveling on a boat that is moving at speed v relative to the penguin on shore at a nearby island. What speed u does the penguin observe the car go? The Galilean answer is the common sense one: u' + v. But this is wrong according to relativity when high speeds are present.



PC1 (Practice Problem). Galilean Velocity Addition. Start with

$x' = x - vt$ and $t' = t$. Set $u' = \dfrac{\Delta x'}{\Delta t'}$ and $u = \dfrac{\Delta x}{\Delta t}$. Show $u = u' + v$.

PC2 (Practice Problem). Relativistic Velocity Addition. Start with

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad t' = \frac{t - x\frac{v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{or} \quad x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad t = \frac{t' + x'\frac{v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Set $u' = \dfrac{\Delta x'}{\Delta t'}$ and $u = \dfrac{\Delta x}{\Delta t}$. Show that $u = \dfrac{u' + v}{1 + \frac{u'v}{c^2}}$.

Then define $\beta_o = \dfrac{v}{c}$, $\beta = \dfrac{u}{c}$, and $\beta' = \dfrac{u'}{c}$ to show $\beta = \dfrac{\beta' + \beta_0}{1 + \beta'\beta_0}$.

Note that if you relativistic ally add c and c, you get c. Let's do it.

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}}$$

$$u = \frac{c + c}{1 + \frac{cc}{c^2}} = \frac{2c}{1+1} = c$$

You just can't go faster than the speed of light. Otherwise, you would violate Postulate 2. Postulate 2 states you must always measure the speed of light as the speed of light c no matter how fast you are going when the light zips by you.

PC3 (Practice Problem). Perpendicular Relativistic Velocity Addition. Start with

Derive the relativistic formula for the perpendicular velocity transformation where the object moves in the K' frame with speed $\vec{u}' = (u_x', u_y')$ and the K frame measures $\vec{u} = (u_x, u_y)$. The K' frame moves at the usual speed v in the x-direction relative to the K frame. Your answer will be

$$u_y = \frac{u_y' \sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{u_x' v}{c^2}}.$$

Hint: This one is done just like the previous one. You set up $u_y = \frac{\Delta y}{\Delta t}$ and then use the Lorentz transformation to replace the Δy land Δt in terms of $\Delta y'$, $\Delta x'$, and $\Delta t'$. Note since K' moves only along the x-axis, you have $y' = y$ and therefore $\Delta y' = \Delta y$, which helps you out. Did you figure out for the previous problem that in situations like these you have to divide top and bottom by $\Delta t'$?

C6. Four-Vectors

The world is four dimensional. You need three spatial coordinates and one time coordinate. So we make a vector with four components where we put time first. But we need to have the same dimension for each coordinate so we use ct instead of t.

$$x^\mu = (ct, x, y, z) \text{ and } dx^\mu = (cdt, dx, dy, dz)$$

It is traditional to use an upper Greek letter for the four-vector and Roman letters for the spatial dimensions: $x^i = (x, y, z)$, where i = 1, 2, and 3 for x, y, and z respectively. Note that $\mu = 0$ refers to time. The four coordinates are labeled then as follows:

$$x^0 = ct \text{, } x^1 = x \text{, } x^2 = y \text{, and } x^3 = z.$$

You know by the context whether a superscript means component of a four-vector or taking a variable to a power. You might say we should use subscripts. We can't because subscripts are defined in this way:

$$x_\mu = (ct, -x, -y, -z) \text{ and } dx_\mu = (cdt, -dx, -dy, -dz).$$

This is done so that the differential length in four-dimensional spacetime is defined as $ds^2 = \sum_{\mu=0}^{3} dx^\mu dx_\mu = dx^0 dx_0 + dx^1 dx_1 + dx^2 dx_2 + dx^3 dx_3 \equiv dx^\mu dx_\mu$, where you can omit the summation sign. This is Einstein's summation convention: the sum is understood. We get

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2.$$

This is invariant. If we put an object in a primed frame and do not let it move around in that frame, then we have dx' = dy' = dz' = 0. We can then write

$$ds^2 = c^2 (dt')^2 - (dx')^2 - (dy')^2 - (dz')^2 = c^2 (dt')^2 \text{, and this must equal}$$

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2.$$

Therefore, $ds^2 = c^2 (dt')^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$. The differential time interval $dt' \equiv d\tau$ is called the proper time since it is the time of the clock in its own frame.

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \,,$$

Therefore,

$$c^2 d\tau^2 = c^2 \left[1 - \frac{1}{c^2}\frac{dx^2}{dt^2} - \frac{1}{c^2}\frac{dy^2}{dt^2} - \frac{1}{c^2}\frac{dz^2}{dt^2} \right] dt^2 = c^2 \left[1 - \frac{v^2}{c^2} \right] dt^2$$

A compact result is

$$d\tau = \sqrt{1 - \frac{v^2}{c^2}}\, dt \,.$$

Do you recognize this? How about in the following form?

$$dt = \frac{d\tau}{\sqrt{1 - \frac{v^2}{c^2}}}$$

It is the differential form of your time-dilation formula.

Now we define a special four-velocity u^μ by differentiating our four-vector by the proper time. Here is our four-vector:

$$x^\mu = (ct, x, y, z) \,.$$

Take the derivative with respect to τ we get

$$u^\mu \equiv \frac{dx^\mu}{d\tau} = \left(c\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\left(c\frac{dt}{dt}, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

$$u^\mu = \frac{dx^\mu}{d\tau} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}(c, v_x, v_y, v_z)$$

$$u^{\mu} = \frac{dx^{\mu}}{d\tau} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}(c, \vec{v})$$

It is common practice to define gamma such that

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} .$$

Then we have

$$u^{\mu} = \frac{dx^{\mu}}{d\tau} = \gamma(c, \vec{v})$$

You are free to roam in space and have speed $\vec{v}$ in the spatial dimensions but you are force to be pulled forward in the time dimension at the speed of light since $u^0 = c$.

The four-momentum is defined by multiplying the four-velocity by the mass m.

$$p^{\mu} \equiv m\frac{dx^{\mu}}{d\tau} = \gamma(mc, m\vec{v})$$

Einstein was led to using the special four-momentum to replace the classical p = mv since by taking the derivative with respect to the proper time you have no ambiguity in the time. The proper time is unique, the value the clock reads in the frame it is in. The "length" of each four-vector is invariant from frame to frame.

The four-vectors $x^{\mu} = (ct, \vec{r})$ and $p^{\mu} = \gamma(mc, m\vec{v})$ have lengths

$$x^{\mu}x_{\mu} = c^2t^2 - r^2 = \tau^2$$

$$p^{\mu}p_{\mu} = \gamma^2\left[m^2c^2 - m^2v^2\right] = \frac{1}{1-\frac{v^2}{c^2}}m^2c^2\left[1-\frac{v^2}{c^2}\right] = m^2c^2$$

The first gives the proper time squared as the invariant and the second one is the mass squared as the invariant with the speed of light squared in there too.

C7. Work and Energy

In classical physics $F = ma$, which we can write as $F = m\frac{dv}{dt}$. Since the momentum in classical physics is $p = mv$, we can also write $F = \frac{dp}{dt}$. This last form is actually the best form for Newton's second law since it allows for rocket problems where mass is shot out the back and thus m changes as well as velocity v. Of course in its formal form it is a vector equation $\vec{F} = \frac{d\vec{p}}{dt}$.

Let's push or pull a mass from rest in outer space (no friction) through some distance x. The work you do is defined as the product of the force component along the direction of motion times the distance. Think about this definition.

Work = Force x Distance = Fd

If you were paying workers to push boxes of appliances on rollers across a warehouse floor, doesn't it make sense to pay them depending on how much force they apply times the distance? Would you pay anything if someone was applying a force to a refrigerator against the wall going nowhere? Then the distance would be zero and force times distance is zero, i.e., no work. What about the zombie who walks around with hands extended applying zero force to the air? The zombie pushes nothing. So force times distance is zero due to the zero force. What about someone lying down (F = 0, d = 0)?

PE4 (Practice Problem). Let y = u^2 and u = 3x. Find y = y(x) and dy/dx. Now calculate dy/dx from the chain rule (dy/du)(du/dx). Do your answers agree?

To allow for non-constant situations we use the calculus definition of applying a force F for an infinitesimal distance dx and do an integral. We will take the force aligned with x and write

$$W = \int F(x)dx .$$

Lets apply a force to a mass in outer space from rest and then let go of it. What is the work we do?

$$W = \int F\,dx = \int ma\,dx = \int m\frac{dv}{dt}\,dx = \int m\frac{dv}{dx}\frac{dx}{dt}\,dx = \int_0^v mv\,dv = \frac{1}{2}mv^2$$

Do you like our tricks with the chain rule, differentials, and changing the integration variable? This result is called the work-energy theorem. You do work on the mass and this work goes into energy in the form of motion. So we define the energy due to the work we did as the kinetic energy since the mass is now moving through space.

$$E = \frac{1}{2} m v^2 .$$

If you apply a force on an already moving object, the work you do is expressed as a difference of kinetic energies. See below for the more general case of the work-energy theorem.

$$W = \int_{x_1}^{x_2} F\, dx = \int_{v_1}^{v_2} m v\, dv = \frac{1}{2} m {v_2}^2 - \frac{1}{2} m {v_1}^2$$

Let's play the same game for the relativistic case, where the new momentum is taken. The relativistic case is worked out for you in the video lecture using the conventional approach. I an supplemental video we show you a shortcut and include it here in our text. I learned about it from Joe Heafner (Catawba Valley Community College, Hickory, NC), who presented this derivation due to Holladay, at our Fall 2021 North Carolina Section of the American Association of Physics Teachers Meeting. The reference is Wendell G. Holladay, "The derivation of relativistic energy from the Lorentz γ," *American Journal of Physics* **60**, 281 (1991).

We start again with $W = \int F(x) dx$ where the work gives the change in energy $W = \Delta E$ via the work-energy theorem. For an infinitesimal amount of work we can write

$$dE = F dx .$$

We introduce Newton's Second Law at this stage. $F = \frac{dp}{dt}$, arriving at

$$dE = \frac{dp}{dt} dx .$$

Now mathematicians get nervous when physicists start moving differentials around, so we will ease up and use finite deltas:

$$\Delta E = \frac{\Delta p}{\Delta t} \Delta x .$$

Now we can rearrange things safely,

$$\Delta E = \Delta p \frac{\Delta x}{\Delta t}$$

and then proceed to take the limit to find the differential form

$$dE = dp \frac{dx}{dt}.$$

Substituting $v = \dfrac{dx}{dt}$, we have $dE = (dp)(v)$ and

$$dE = vdp.$$

For the shortcut derivation we use our gamma formula

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Write this relation in the form

$$\gamma \sqrt{1 - \frac{v^2}{c^2}} = 1.$$

Square both sides,

$$\gamma^2 (1 - \frac{v^2}{c^2}) = 1,$$

$$\gamma^2 - \frac{(\gamma v)^2}{c^2} = 1.$$

Now the magic begins. We take a differential of each side.

$$d(\gamma^2) - d\left[\frac{(\gamma v)^2}{c^2}\right] = 0$$

Using the rules of differential calculus

$$2\gamma d(\gamma) - \frac{2(\gamma v)}{c^2} d(\gamma v) = 0 \, ,$$

and

$$2\gamma d(\gamma) = \frac{2(\gamma v)}{c^2} d(\gamma v) \, .$$

Multiply both sides by $\dfrac{mc^2}{2\gamma}$, giving

$$mc^2 d(\gamma) = mvd(\gamma v) \, .$$

The next step is to move constants into the differentials

$$d(\gamma mc^2) = vd(\gamma mv) \, .$$

Now compare with

$$dE = vdp \, .$$

We have derived in one stroke both the relativist energy and momentum formulas:

$E = \gamma mc^2$ and $p = \gamma mv$, i.e.,

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{and} \quad p = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}} \, .$$

PC5 (Practice Problem). Derivation of Einstein's Most Famous Equation the Conventional Way. Some steps are worked out for you. Details are found in the video lecture for this class.

Start with our four-momentum $p^\mu \equiv m\dfrac{dx^\mu}{d\tau} = \gamma(mc, m\vec{v})$ and from this obtain

$$\vec{p} = m\frac{d\vec{r}}{d\tau} = \gamma m\vec{v} .$$

Then, the work on pushing a mass in outer space (which eliminates friction) is

$$W = \int F\,dx = \int \frac{dp}{dt}\,dx = \int \frac{d}{dt}\left(\frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}\right)dx .$$

Use chain-rule and differential tricks similar to those that we did for the classical case to derive the following equivalent form for the work.

$$W = \int \frac{d}{dv}\left(\frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}\right) v\,dv$$

Now we will use the integration-by-parts trick. You can remember this trick from the product rule of differentiation given below.

$$\frac{d(fg)}{dv} = g\frac{df}{dv} + f\frac{dg}{dv}$$, which is also $$g\frac{df}{dv} = \frac{d(fg)}{dv} - f\frac{dg}{dv} .$$

So you can move the derivative off the g here and move it to the f. Use this trick to show that

$$W = \int \frac{d}{dv}\left(\frac{mv^2}{\sqrt{1-\frac{v^2}{c^2}}}\right) dv - \int \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}\,dv .$$

Note that your first integral is no problem as you have a perfect differential. Do the second integral using conventional methods. Do not look this integral up as you can do it easily with conventional methods. Put in your limits of integration which we take from speed 0 to some speed v as we apply the force to the mass initially at rest. Show that you get

$$W = \left[\frac{mv^2}{\sqrt{1-v^2/c^2}} + mc^2\sqrt{1-v^2/c^2} \right]_0^v .$$

Work this out to arrive at,

$$W = \frac{mv^2}{\sqrt{1-v^2/c^2}} + mc^2\sqrt{1-v^2/c^2} - mc^2$$ and continue the steps to get

$$W = \frac{mc^2}{\sqrt{1-v^2/c^2}} - mc^2$$, which is also $W = E(v) - E(v=0)$.

Therefore,

$$E = \frac{mc^2}{\sqrt{1-v^2/c^2}}$$

End of Practice Problem PC4.

Note immediately that if the speed $v = 0$, your expression

$$E = \frac{mc^2}{\sqrt{1-v^2/c^2}}$$ reduces to $E = mc^2$.

You still have energy, $E = mc^2$, when the mass is at rest. This is called the rest energy. The kinetic energy KE according to Einstein is then the total energy for a moving mass minus the energy it has when it is at rest.

$$KE = \frac{mc^2}{\sqrt{1-v^2/c^2}} - mc^2$$

PC6 (Practice Problem). **Correspondence Limit.** Show that a Taylor Series expansion on the first term in $KE = \dfrac{mc^2}{\sqrt{1 - v^2 / c^2}} - mc^2$ will lead to the result that at slow speeds Einstein's relativistic version of the work-energy theorem reduces to Newton's: $E = \dfrac{1}{2}mv^2$.

C8. Four-Momentum Revisited

We saw earlier that the four-momentum is

$$p^\mu = m\frac{dx^\mu}{d\tau} = \gamma(mc, m\vec{v}).$$

Check out $p^0 = \gamma mc$, What's this? Well, you just discovered in the previous section that $E = \dfrac{mc^2}{\sqrt{1 - v^2 / c^2}}$, which is the same as $E = \gamma mc^2$. So that means $p^0 = \dfrac{E}{c}$. This is elegant, the fourth dimension here, extending our concept of momentum is related to the energy. Four dimensions in spacetime means space and time while four dimensions in the momentum arena means momentum and energy.

Summary: $x^\mu = (ct, \vec{r})$, $u^\mu = \dfrac{dx^\mu}{d\tau} = \gamma(c, \vec{v})$

$p^\mu = (\dfrac{E}{c}, \vec{p})$, where $\vec{p} = \dfrac{m\vec{v}}{\sqrt{1 - \dfrac{v^2}{c^2}}}$

In the rest frame, $p^\mu = (\dfrac{E}{c}, \vec{0})$. Think of Energy / c as "momentum in time."

Review of our invariant equations. We found earlier that

$$p^\mu p_\mu = m^2c^2$$

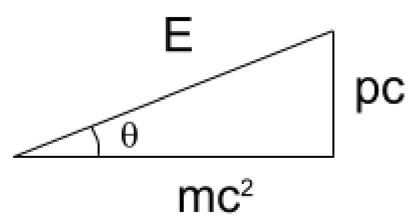
With $p^{\mu} = (\frac{E}{c}, \vec{p})$, we must have

$$p^{\mu} p_{\mu} = \frac{E^2}{c^2} - p^2 = m^2 c^2 ,$$

which we can write as

$$E^2 = m^2 c^4 + p^2 c^2 ,$$

and we can remember this by a cute right triangle.



At rest, the angle is zero and $E = mc^2$.

At the relativistic extreme the angle is 90° and $E = pc$, which is true for light.