Theoretical Physics
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Chapter A Solutions. Taylor Series, Rotation Matrix, Groups

HW-A1. Taylor Series. Find $f(x) = \cos(x)e^x$ to cubic power in x two ways.

a) Method 1. The usual Taylor Series.

$$f^{(0)} = \cos(x)e^x$$

$$f^{(1)}(x) = \cos(x)e^x - \sin(x)e^x$$

$$f^{(2)}(x) = \cos(x)e^x - \sin(x)e^x - \sin(x)e^x - \cos(x)e^x = -2\sin(x)e^x$$

$$f^{(3)}(x) = -2\sin(x)e^x - 2\cos(x)e^x$$

$f^{(0)} = \cos(x)e^x$	$f^{(0)}(0) = 1$	$f(0) = 1$
$f^{(1)}(x) = \cos(x)e^x - \sin(x)e^x$	$f^{(1)}(0) = 1$	$f^{(1)}(0)x = x$
$f^{(2)}(x) = -2\sin(x)e^x$	$f^{(2)}(0) = 0$	$f^{(2)}(0)\frac{x^2}{2!} = 0$
$f^{(3)}(x) = -2\sin(x)e^x - 2\cos(x)e^x$	$f^{(3)}(0) = -2$	$f^{(3)}(0)\frac{x^3}{3!} = -\frac{x^3}{3}$

Now add that last column so that

$$f(x) = f^{(0)}(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$ becomes

$$\boxed{f(x) = \cos(x)e^x = 1 + x - \frac{1}{3}x^3 + \ldots}$$

b) Method 2. Multiplying Taylor Series. $f(x) = \cos(x)e^x$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$ and $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}.+..$$

To cubic in x: $$f(x)=\cos(x)e^{x}=(1-\frac{x^{2}}{2!}+\ldots)(1+x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\ldots)$$

$$f(x)=\cos(x)e^{x}=(1)(1+x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!})-\frac{x^{2}}{2!}(1+x)+\ldots$$

$$f(x)=\cos(x)e^{x}=1+x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}-\frac{x^{2}}{2!}-\frac{x^{3}}{2!}+\ldots$$

$$f(x)=\cos(x)e^{x}=1+x+\frac{x^{3}}{3!}-\frac{x^{3}}{2!}+\ldots$$

$$f(x)=\cos(x)e^{x}=1+x+(\frac{1}{6}-\frac{1}{2})x^{3}+\ldots$$

$$f(x)=\cos(x)e^{x}=1+x+(\frac{1-3}{6})x^{3}+\ldots$$

$$f(x)=\cos(x)e^{x}=1+x-\frac{2}{6}x^{3}+\ldots$$

$$\boxed{f(x)=\cos(x)e^{x}=1+x-\frac{1}{3}x^{3}+\ldots}$$

HW-A2. Matrix Multiplication. Consider the Pauli matrices.

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

a) $$\sigma_x \sigma_y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0\cdot 0 + 1\cdot i & 0\cdot(-i) + 1\cdot 0 \\ 1\cdot 0 + 0\cdot i & 1\cdot(-i) + 0\cdot 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$\sigma_x \sigma_y = i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Problem asks for in terms of the sigmas. So there is one final step.

$$\boxed{\sigma_x \sigma_y = i\sigma_z}$$

b) $$\sigma_y \sigma_x = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0\cdot 0 + (-i)\cdot 1 & 0\cdot 1 + (-i)\cdot 0 \\ i\cdot 0 + 0\cdot 1 & i\cdot 1 + 0\cdot 0 \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

$$\sigma_y \sigma_x = -i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Problem asks for in terms of the sigmas. So there is one final step. $\boxed{\sigma_y \sigma_x = -i\sigma_z}$

c) $[\sigma_x, \sigma_y] \equiv \sigma_x \sigma_y - \sigma_y \sigma_x$? From a) and b) we have $\sigma_x \sigma_y = i\sigma_z$ and $\sigma_y \sigma_x = -i\sigma_z$. Therefore, $[\sigma_x, \sigma_y] = \sigma_x \sigma_y - \sigma_y \sigma_x = i\sigma_z - (-i\sigma_z) = 2i\sigma_z$

$$\boxed{[\sigma_x, \sigma_y] = 2i\sigma_z}$$

d) $\{\sigma_x, \sigma_y\} \equiv \sigma_x \sigma_y + \sigma_y \sigma_x$? Using again the results from a) and b), we get zero.

$$\boxed{\{\sigma_x, \sigma_y\} = 0}$$

HW-A3. Rotation Matrix. Show that

$$R(30°)=\begin{bmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix},\quad R(60°)=\begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix},\ \text{and}\ R(90°)=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}.$$

Then by explicit matrix multiplications, show that

$$R(90°)=R(30°)R(60°)=R(60°)R(30°)\,.$$

Solution

$$R(\theta)=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

$$R(30°)=\begin{bmatrix}\cos 30° & \sin 30°\\ -\sin 30° & \cos 30°\end{bmatrix}=\begin{bmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}$$

$$R(60°)=\begin{bmatrix}\cos 60° & \sin 60°\\ -\sin 60° & \cos 60°\end{bmatrix}=\begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}$$

$$R(90°)=\begin{bmatrix}\cos 90° & \sin 90°\\ -\sin 90° & \cos 90°\end{bmatrix}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$$

$$R(30°)R(60°)=\begin{bmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}\begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}$$

$$R(30°)R(60°)=\begin{bmatrix}\frac{\sqrt{3}}{2}\frac{1}{2}-\frac{1}{2}\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2}+\frac{1}{2}\frac{1}{2}\\ -\frac{1}{2}\frac{1}{2}-\frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2} & -\frac{1}{2}\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}\frac{1}{2}\end{bmatrix}$$

$$R(30°)R(60°)=\begin{bmatrix}0 & \frac{3}{4}+\frac{1}{4}\\ -\frac{1}{4}-\frac{3}{4} & 0\end{bmatrix}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}=R(90°)$$

$$R(60°)R(30°)=\begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}$$

$$R(60°)R(30°)=\begin{bmatrix}\frac{1}{2}\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}\frac{1}{2} & \frac{1}{2}\frac{1}{2}+\frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2}-\frac{1}{2}\frac{1}{2} & -\frac{\sqrt{3}}{2}\frac{1}{2}+\frac{1}{2}\frac{\sqrt{3}}{2}\end{bmatrix}$$

$$R(60°)R(30°) = \begin{bmatrix} 0 & \frac{1}{4}+\frac{3}{4} \\ -\frac{3}{4}-\frac{1}{4} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = R(90°)$$

HW-A4. A Three-Element Group. Show that the set $S = \{R(0°), R(120°), R(-120°)\}$ forms a group under multiplication. Construct the multiplication table.

a) Closure. Show that closure is met by constructing the complete multiplication table for the group. Your multiplication table will have 9 entries where each of these entries will be one of the three elements from the set: $R(0°)$, $R(120°)$, or $R(-120°)$.

Photo Courtesty Asheville Ballet, Director Ann Dunn.

Use dancer turns to construct the multiplication table.

	$R(0°)$	$R(120°)$	$R(-120°)$
$R(0°)$	$R(0°)$	$R(120°)$	$R(-120°)$
$R(120°)$	$R(120°)$	$R(-120°)$	$R(0°)$
$R(-120°)$	$R(-120°)$	$R(0°)$	$R(120°)$

b) Association.

Method 1 Dancing Again! Consider a dancer turning through angles α , β , and γ by doing the later two first and comparing to the result when the first two are done first.

$$R(\alpha)\cdot[R(\beta)\cdot R(\gamma)] = R(\alpha)\cdot R(\beta+\gamma) = R(\alpha+\beta+\gamma)$$

$$[R(\alpha)\cdot R(\beta)]\cdot R(\gamma) = R(\alpha+\beta)\cdot R(\gamma) = R(\alpha+\beta+\gamma)$$

$$R(\alpha)\cdot[R(\beta)\cdot R(\gamma)] = [R(\alpha)\cdot R(\beta)]\cdot R(\gamma)$$

Method 2. Matrix Representation. A rotation can be represented by a matrix as

$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

Two rotations are then represented by matrix multiplication.

Since matrix multiplication associates, association applies to our set.

c) Identity. The identity element is $R(0°)$.

d) Inverse. Below is each element and its inverse. The inverses are also in the set.

$R(0°)$	$R^{-1}(0°) = R(0°)$
$R(120°)$	$R^{-1}(120°) = R(-120°)$
$R(-120°)$	$R^{-1}(-120°) = R(120°)$

$$R(0°)R^{-1}(0°) = R(0°) \equiv I$$

$$R(120°)R^{-1}(120°) = R(120°)R(-120°) = R(0°) = I$$

$$R(-120°)R^{-1}(-120°) = R(-120°)R(120°) = R(0°) = I$$

e) Abelian? Yes, the group is abelian – from the multiplication table. The off diagonal elements are equal, which means you can apply the binary operation for two of the elements in any order.

	$R(0°)$	$R(120°)$	$R(-120°)$
$R(0°)$	$R(0°)$	$R(120°)$	$R(-120°)$
$R(120°)$	$R(120°)$	$R(-120°)$	$R(0°)$
$R(-120°)$	$R(-120°)$	$R(0°)$	$R(120°)$