OPEN ACCESS

PAPER

Phys. Educ. **59** (2024) 015032 (8pp) iopscience.org/ped

Skyline horizon from a mountain 130 km away

Michael J Ruiz

Department of Physics and Astronomy, University of North Carolina at Asheville, Asheville, NC 28804, United States of America

E-mail: mjtruiz@gmail.com

Abstract

How far is the horizon? A photograph of a city skyline taken from a mountain (elevation 1610 m) at a distance of 130 km is used to introduce the question. The formula that relates the distance to the horizon from an elevated observer is easy to derive when refraction is neglected. The observed horizon distance of the skyline is compared with the theoretical horizon limit of this simple formula. Finally, refraction is discussed and shown to increase the limit by about 10%. A highly interactive component making measurements on Google Maps is included. Introductory students at the high school and first-year college level will find this real-life physics application that incorporates photography, geography, online maps, Earth science, and elementary mathematics very stimulating.

Keywords: horizon, refraction, geometrical optics, atmospheric refraction, terrestrial refraction, geometry

1. Location of observer

The inspiration for this paper comes from a beautiful photograph of the Charlotte skyline 130 km away, taken by Hugh MacRae Morton from near the Mile High Swinging Bridge on Grandfather Mountain, both in North Carolina, USA. The 69.5 m (228 ft) suspension bridge has an elevation above sea level of one mile (1609 m) and spans a chasm 24 m (80 ft) deep.

Morton (1921–2006) was a 'developer, conservationist and photographer' [1], who owned Grandfather Mountain at the time he took the photo. Morton, having inherited the mountain from his grandfather in 1952, 'never considered himself its 'owner', but rather its legal guardian' [2]. He developed the mountain over the years, making it available to the public. Morton's ancestors, the MacRaes, were Highland Scots that came to North America 'from Scotland's western coast in 1770' [3]. See figure 1 for a 2023 photo of the swinging bridge.

2. The skyline

The focus of this paper is a photograph of the Charlotte skyline taken far away from Grandfather Mountain [4, 5]. The Charlotte skyline close to the city appears in a 2023 photograph in figure 2.

Figure 3 is the photo by Hugh Morton of the Charlotte skyline from Grandfather Mountain.

Figure 1. Mile High Swinging Bridge, Grandfather Mountain, Linville, North Carolina, USA. Photo by Author in Fall 2023.

Figure 2. The Charlotte Skyline, Charlotte, North Carolina, USA, in 2023. Photo by Author.

For the skyline to be visible at this distance, the weather has to be very clear. Here is the photographer's comment on his photo [4].

> My favourite picture of the Charlotte skyline was made from a spot near the Mile High Swinging Bridge on Grandfather Mountain. According to FAA air maps, Charlotte is 87 air miles from Grandfather. I took the picture in mid-December when a cold front had just cleared the air. It would be exciting to have air quality like this every day, but unfortunately this sort of visibility is rare. [Hugh Morton]

Waves of mountainous hills appear between the observer and the distant Charlotte skyline in the photo of figure 3. There are fewer buildings in this photo when compared to figure 2 since the Morton photo was taken much earlier. From the completed dates of the buildings [6] and the publication of the photo in Morton's 2003 book [4], the date of the photo is bracketed between 1992 and 2003. The Charlotte Museum of History gives the date as December 1993 [7]. The original is a 35 mm colour roll film negative (24 $\times$ 36 mm) in the Wilson Special Collections Library, the University of North Carolina at Chapel Hill, Chapel Hill, North Carolina, USA [5].

3. Google Maps and the skyline distance

The power of using online Google mapping tools in physics instruction has been demonstrated by Vollmer in this journal [8]. Since 'Google Maps' was not available in the 1990s, Morton consulted the Federal Aviation Administration (FAA) air maps to arrive at his estimate of 87 miles (140 km) to the Charlotte skyline from Grandfather Mountain. In this section we show how students can make their own improved measurements using Google Maps. It is fun.

Go to google.com/maps. For directions, click on the small blue diamond with a bent arrow near where it says 'Search Google Maps'. Enter 'Grandfather Mountain North Carolina' for the starting point. Then enter 'Charlotte North Carolina' for the destination. You will see the suggested route to be taken by car. We want the direct distance, i.e. 'as the crow flies'. So right click in the middle of the small circle at the starting point and choose 'Measure distance'. Then click in the centre of the small circle at the destination. The measurement should appear. My measurement to two significant figures was 130 km (81 mi) each time I did it. If you use 'Mile High Swinging Bridge, Blowing Rock Highway' as the starting point, you find the same result to two significant figures. Measuring each of these two cases a few times helps students appreciate the importance of significant figures.

4. The simple theoretical horizon formula

The simple formula neglecting refraction for the theoretical horizon distance is very easy to derive

Figure 3. View of Charlotte 130 km away from Grandfather Mountain, Hugh Morton Photographs and Films, Wilson Special Collections Library, the University of North Carolina at Chapel Hill, Chapel Hill, USA. The mountainous terrain appears as waves of hills and valleys. Reproduced with permission from [5].

and serves as a 'standard problem in elementary physics or mathematics' [9]. The analysis is based on the right triangle of figure 4 and can be traced back to Snell of refraction law fame [10]. The elevation height is h, the radius of the Earth is R, the direct distance from the observer on the mountain to the horizon is d, and the distance from the base of the mountain to the horizon along the circumference of the Earth is the arclength s.

At the horizon point, the angle between the line d and the slanted radius R in the diagram is 90°. Therefore, we have a right triangle with sides d, R, and hypotenuse $R+h$. Using the Pythagorean formula,

$$R^2 + d^2 = (R+h)^2. \tag{1}$$

Working out the right side,

$$R^2 + d^2 = R^2 + 2Rh + h^2, \tag{2}$$

leading to

$$d^2 = 2Rh + h^2. \tag{3}$$

The radius of the Earth is much greater than the height of the mountain. Therefore, $R >> h$ and $2Rh >> h^2$. If students are seeing this kind of analysis for the first time, one can insert the values to justify neglecting h^2. The height $h = 1.610$ km and the authalic[1] radius of the Earth is $R = 6371$ km. Comparing h^2 to $2Rh$ then gives

$$\frac{h^2}{2Rh} = \frac{h}{2R} = \frac{1.61}{2\,(6371)} = 1.3 \times 10^{-4}, \tag{4}$$

i.e. h^2 is one ten-thousandth of $2Rh$. So we can write equation (3), neglecting h^2, as

$$d^2 = 2Rh. \tag{5}$$

Taking the square root, we arrive at the very simple formula

$$d = \sqrt{2Rh}. \tag{6}$$

[1] The Earth is a slightly flattened sphere since it bulges at the equator due to its rotation. The authalic radius of the Earth is the radius of the sphere with the same total surface area of the Earth. It gives a nice global average.

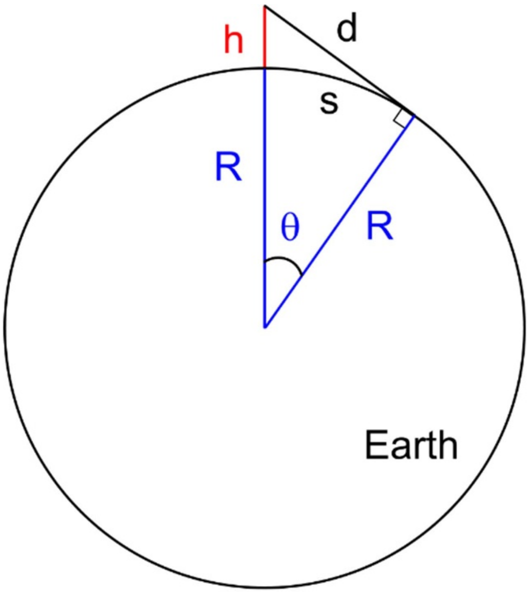

Figure 4. Diagram relevant for calculating the geometric distance to the horizon from an observer on a mountain with elevation *h*.

Finally, the teacher can show that the direct distance $d \approx s$ by having students sketch the diagram closer to scale, i.e. a much larger circle with a much smaller angle θ than shown in figure 4. Or one can resort to trigonometry using $\theta \approx \tan\theta$ for small angles in radians and the fact that $\tan\theta = d/R$ from figure 4. Therefore,

$$s = R\theta \approx R\tan\theta = R\frac{d}{R} = d. \quad (7)$$

Students are encouraged to use their calculators to examine how large one can make the angle and still be within a good approximation. For example, considering an angle that is not so small, such as $10° = 0.17$ rad, one finds that the approximation is still very good: $\tan 0.17 = 0.17$, to two significant figures. But this angle corresponds to an arclength well beyond what is needed for our much shorter distance. The arclength for $10°$ is $s = R\theta = (6371 \text{ km})(0.17) = 1080$ km, a value about eight times the distance required for our case where d is only 130 km. Therefore, students should feel comfortable with the approximation in equation (7), even if they are not familiar with the expansion of the tangent function.

Wickramasinghe [11] has pointed out that when he asks his students about the horizon distance, he gets answers from 1 to 100 miles (1.6–160 km). He expresses equation (6) with a units trick that dates back to at least the 1870s [12]. The aim is to obtain a formula when one can insert the height *h* in metres (or feet) and obtain the answer in kilometres (or miles):

$$d(\text{in km}) = \sqrt{2R(\text{in km})h(\text{in km})}, \quad (8a)$$

$$d(\text{in km}) = \sqrt{2(6371 \text{ km})}\sqrt{\frac{h(\text{in m})}{1000}}, \quad (8b)$$

$$d(\text{in km}) = 3.57\sqrt{h(\text{in m})}. \quad (8c)$$

We are now ready to calculate an estimate for the theoretical horizon limit for our elevation of 1610 m, neglecting refraction. But we need to check the elevation of Charlotte. In a browser, go to freemap.tools.com and choose 'Elevation Finder.' In the 'Find a Location' search bar, type Charlotte, North Carolina. You will find 230 m to two significant figures. Since Charlotte has an elevation of 230 m, we should use 1610–230 m = 1380 m for the height. The result is

$$d(\text{in km}) = 3.57\sqrt{1380(\text{in m})} = 133 \text{ km}, \quad (9)$$

which corresponds to our actual distance of 130 km to two significant figures. We are at the geometric horizon limit, which will make seeing surface features difficult. But since we are looking at skyscrapers, our geometrical analysis alone explains the sighting.

4.1. The skyline angle subtended to the naked eye

The angle subtended by the skyline visible to the naked eye should be determined since photographers will often zoom in or employ a telephoto lens for a photo like the one in figure 3. We can obtain a good estimate of the skyline's subtended angle from Grandfather Mountain since the buildings can be identified by name from [6]. Then, by googling 'tallest buildings in Charlotte' we can obtain the street addresses for the buildings. The main relevant distance is the distance from the left edge of the far-left building to the right edge of the far-right building in figure 3. This distance can be measured on Google Maps when one zooms into centre city.

Remember that the photo in figure 3 was taken in 1993. Therefore, only six skyscraper buildings from [6] need to be considered, the buildings constructed prior to 1993. The left building in figure 3 is the Bank of America Corporate Center (completed in 1992), the tallest building in Charlotte with a height of 265 m and located at 100 N Tryon Street. The building at the far right is 400 South Tryon (1974), named after its address. But since Google Maps does not clearly label building names, it is best to find directions from 100 N Tryon to 400 South Tryon and measure the distance between the start and end points. You will find 0.5 km. However, I carefully measured the distance from the far edges of the buildings on the aerial view of Google Maps and found a more precise distance of 0.58 km. Therefore, the angle subtended from a distance of 130 km is

$$\phi = \frac{0.58\ \text{km}}{130\ \text{km}} \cdot \frac{180^\circ}{\pi} = 0.26^\circ. \qquad (10)$$

Since the angle subtended by the diameter of the Moon in the night sky is 0.5°, the Charlotte skyline subtends an angle, as viewed from Grandfather Mountain with the naked eye, about one-half that of the Moon's diameter. The silhouettes of the buildings are discernible against the bright sky when the weather is exceptionally clear and one is viewing near dawn.

The above analysis integrates geography, maps, geometry, and student measurements, illustrating the interdisciplinary power of physics. One can even measure the subtended angle for viewing the photo in this journal or on a computer monitor, depending on the reader's viewing distance. On my large monitor the building separation distance is 2 cm as I sit 100 cm from the screen. The angle is

$$\phi_{\text{my monitor}} = \frac{2\ \text{cm}}{100\ \text{cm}} \cdot \frac{180^\circ}{\pi} \approx 1^\circ. \qquad (11)$$

5. Refraction

Though our geometrical analysis explains visibility of the Charlotte skyline from Grandfather Mountain, we include a section on refraction mainly as background material for the teacher. However, the discussion has no calculus. Therefore, it can be presented to introductory students. For those interested in derivations, [10] is an excellent source.

In the troposphere, the lowest part of the atmosphere reaching up to about 10 km, air becomes less dense and cooler with increasing altitude. The result is a decreasing index of refraction with height. However, at times temperature inversions can occur, which changes things.

Under normal conditions, the trajectory of a light ray will curve with its concave side facing the Earth. In 1977, Johnston [13] published a nice demonstration to illustrate this arc. He developed the demonstration for his high school students by setting up a vertical density gradient in a water tank. He used sugar to form layers of different sugar concentrations [13]. Then a laser beam dramatically revealed a curved light path seen from the side due to scattering.

The horizon 'problem is no longer so simple if one takes into account the fact that the atmosphere is a refracting medium whose index of refraction decreases progressively with height' [9]. The horizon formula with refraction appeared in a 1913 letter to the editor in *Nature* by Ball [14], where he indicated that the refraction formula was derived in *Handbuch der Vermessunskunde* by Jordan [15], a book published in 1888.

Ball gives the following formula for the distance to the visible horizon, taking into account refraction:

$$d = \sqrt{\frac{2hR}{1-k}}, \qquad (12)$$

where k is called the refraction coefficient. The derivation involves a model where the refracted light path is approximated as an arc of a large circle. The ratio of the Earth's radius to the radius of curvature of this arc of the continuously refracting light ray gives the refraction coefficient [10],

$$k = \frac{R}{R_{\text{light}}}. \qquad (13)$$

Notice that in the limit of no refraction, the light path is linear, i.e. radius $R_{\text{light}} \to \infty$ and the curvature $k \to 0$. In this limit equation (12) reduces to

$$d = \sqrt{\frac{2hR}{1-k}} \rightarrow \sqrt{2hR}, \tag{14}$$

our previous result, which neglects refraction.

The refraction coefficient k varies depending on the location and weather conditions. The parameter k is also called the Gauss coefficient [10] since Gauss worked on this problem in Hannover, deriving a value of 0.13 with an assigned $\pm 25\%$ uncertainty in 1826 [16].

The formula for $1/R_{\text{light}}$, which can be found in *An Introduction to Atmospheric Physics* by Fleagle and Businger [17], is

$$\frac{1}{R_{\text{light}}} = \frac{n-1}{nT}\left(\frac{g}{R_m} - \gamma\right), \tag{15}$$

where n is the index of refraction of air, T is temperature in kelvins, g is the gravitational constant, R_m is the ideal gas constant divided by the molecular mass of air, and γ is the lapse rate, i.e. negative the change in Kelvin temperature per height in kilometres. Standard values for the parameters found in Fleagle and Businger are $n = 1.000\,293$, $T = 288.15$ K (15 °C), $g = 9.81 \text{ms}^{-2}$, and $R_{\text{m}} = 287.1 \frac{\text{J}}{\text{kg}\cdot\text{K}}$ (for dry air since water vapour is usually at 2% in moist air and can be neglected for our estimate). With these values and the radius of the Earth $R_{\text{Earth}} = 6371$ km, we find

$$k = \frac{R}{R_{\text{light}}} = \frac{34.2-\gamma}{154}, \tag{16}$$

where the lapse rate, γ, is to be entered in K km^{-1}. For nice discussions of equation (16) see Young [18] and Vollmer [19].

Note that when the lapse rate $\gamma > 0$, the temperature drops (lapses) as altitude increases. Lapse rate depends on weather conditions and is negative when there is a temperature inversion. The standard lapse rate is $\gamma = 6.5 \text{ K km}^{-1}$, with realistic values up to $\gamma = 10 \text{ K km}^{-1}$ [19]. A theoretical value of $\gamma = 34.2 \text{ K km}^{-1}$ gives $k = 0$ in equation (16). In this case the change in index refraction 'due to pressure drop with height is just compensated by the accompanying temperature drop' [19] and the light does not refract. Large temperature gradients near hot road surfaces produce negative curvatures ($k < 0$) causing light from objects and the sky to curve upward to the observer, who then perceives that water is up ahead on the road, an inferior mirage [20].

The standard lapse rate $\gamma = 6.5 \text{ K km}^{-1}$ leads to a refraction coefficient of 0.18 from equation (16). If we use this value to see how the correction would modify the numerical result of our geometric horizon formula, we just need to evaluate the factor $\sqrt{\frac{1}{1-k}}$ in equation (12):

$$\sqrt{\frac{1}{1-k}} = \sqrt{\frac{1}{1-0.18}} = 1.10. \tag{17}$$

Equation (17) indicates that our original horizon estimate needs to be extended by 10% when refraction is included. Therefore, the geometric model without considering refraction is quite good.

6. Why is the sighting rare?

Earlier in our paper we quoted the photographer Morton [4] stating that he took his photo 'in mid-December when a cold front had just cleared the air. It would be exciting to have air quality like this every day, but unfortunately this sort of visibility is rare.'

Optimum viewing requires air that is very pure and dry as water vapour can produce mist, haze, fog, and clouds. The presence of particles such as dust, pollen, smoke, exhaust particles from combustion, and pollutants, can scatter light and decrease visibility. Ingredients for visibility over long distances include clean air, non-turbulent air, low humidity, good contrast between the objects and background, and observations near dawn [19]. Viewing in the early morning provides for a nice contrast between the skyline buildings and background sky. The air in the early December morning hours in the mountains of North Carolina can be quite cold and still. The cold front that Morton mentioned cleared the air of particles, purifying the air. Since cold air has less humidity than warm air, the early dawn atmosphere after the cold front that Morton described provided for excellent visibility.

Otherwise, the Charlotte skyline is not seen by tourists from Grandfather Mountain. But occasionally a camper or observer captures the skyline when the conditions are just right. You can google 'Charlotte skyline' and 'Grandfather Mountain.'

Choosing images in your Google search, you will find several photos of the skyline similar to the classic one in this paper.

7. Conclusion

An interdisciplinary student activity has been described which involves an artful photograph, physics, algebra, geography, online map measurements, and Earth science. These subjects are tied together with the question: How far is the horizon? Students are shown a photograph where a city skyline appears at a distance of 130 km, but it is left to the students to measure this distance using Google Maps. The teacher can guide students in deriving a formula for the theoretical limit to the horizon without worrying about refraction. When refraction is taken into consideration, the visible horizon increases about 10%.

Students are also exposed to a problem of great interest to surveyors that falls under the topic geodesy. The strong interdisciplinary character of the horizon question with its real-life application of science principles will stimulate student interest in physics. The beautiful photograph of a distant skyline against the early morning red sky will help give the students a lasting memory of the experience.

Finally, from the object distance (130 km), rough width of the building group (0.5 km), and film image size (about 1/15 of 36 mm from the photo), one can use similar isosceles triangles to estimate the focal length (600 mm).

Data availability statement

No new data were created or analysed in this study.

Acknowledgments

I would like to thank my former student Laurel Buffaloe for informing me that the Charlotte skyline can be seen from Grandfather Mountain on exceptionally clear days. I would also like to thank the University of North Carolina at Chapel Hill for giving me permission to use the Hugh Morton photo of the Charlotte skyline and Jason Tomberlin of the Wilson Special Collections Library at the University of North Carolina at Chapel Hill for emailing me a high-resolution image of the original Hugh Morton photo. Finally I would like to thank Christopher Godfrey, Chair of Atmospheric Sciences, University of North Carolina at Asheville for helpful discussions and insights.

ORCID iD

Michael J Ruiz https://orcid.org/0000-0003-0917-1378

Received 30 July 2023, in final form 27 September 2023
Accepted for publication 21 November 2023
https://doi.org/10.1088/1361-6552/ad0e88

References

[1] Photographer and Conservationist Hugh Morton 2016 An official website of the State of North Carolina (NC Department of Natural and Cultural Resources) (available at: www.ncdcr.gov/blog/2017-01-11/photographer-and-conservationist-hugh-morton-0) (Accessed 12 September 2023)

[2] Grandfather Mountain Stewardship Foundation n.d. Hugh Morton (available at: https://grandfather.com/hugh-morton/) (Accessed 12 September 2023)

[3] Johnson R 2016 *Grandfather Mountain: The History and Guide to an Appalachian Icon* (The University of North Carolina Press)

[4] Morton H 2003 *Hugh Morton's North Carolina* (The University of North Carolina Press) p 22

[5] Morton H C 1994 Hugh Morton Collection of Photographs & Films *View of Charlotte from Grandfather Mountain* (North Carolina Collection Photographic Archives, Wilson Library, University of North Carolina) (available at: https://dc.lib.unc.edu/cdm/ref/collection/morton_highlights/id/7396) (Accessed 12 September 2023)

[6] Short J 2019 A breakdown of how the Charlotte skyline has changed *Charlotte Magazine* (January issue). An online version of the article appeared 26 December 2018 (available at: www.charlottemagazine.com/a-breakdown-of-how-the-charlotte-skyline-has-changed/) (Accessed 12 September 2023)

[7] Kiefer H No date Hugh Morton: North Carolina's photographer (available at: https://charlottemuseum.org/learn/articles/hugh-morton-north-carolinas-photographer/) (Accessed 12 September 2023)

[8] Vollmer M 2013 Measuring distances in Google Earth *Phys. Educ.* **48** 145–9

[9] French A P 1982 How far away is the horizon? *Am. J. Phys.* **50** 795–9
[10] Giudici S 2022 A long range sighting and terrestrial refraction *Eur. J. Phys.* **43** 11p
[11] Wickramasinghe T 2000 Trick of the trade: estimating the distance to the horizon *Phys. Teach.* **38** 528
[12] Cole R L 1913 Letters to the editor: distance of the visible horizon *Nature* **92** 425
[13] Johnston J B 1977 Letters to the editor: atmospheric refraction *Phys. Teach.* **15** 308–9
[14] Ball J 1913 Letters to the editor: distance of the visible horizon *Nature* **92** 344
[15] Jordan W 1888 *Handbuch der Vermessungskunde* (Verlag der J. B. Metzlerschen Buchhandlung)
[16] Brunner F K 1984 Overview of geodetic refraction studies *Geodetic Refraction: Effects of Electromagnetic Wave Propagation Through the Atmosphere* ed F K Brunner (Springer-Verlag) ch A, p 2
[17] Fleagle R G and Businger J A 1980 *An Introduction to Atmospheric Physics* 2nd edn (Academic Press) p 356
[18] Young A T 2006 Understanding astronomical refraction *Observatory* **126** 82–115
[19] Vollmer M 2020 Below the horizon—the physics of extreme visual ranges *Appl. Opt.* **59** F11–9
[20] Ruiz M J 2019 Road mirage angle *Phys. Educ.* **54** 9p

Michael J Ruiz is professor emeritus, Department of Physics and Astronomy, at the University of North Carolina at Asheville (UNCA), USA. He received his PhD in theoretical physics from the University of Maryland, USA. His innovative courses with a strong online component aimed at general students have been featured on CNN.