PAPER

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Free sixteen harmonic Fourier series web app with sound

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Abstract

An online HTML5 Fourier synthesizer app is provided that allows students to manipulate sixteen harmonics and construct periodic waves. Students can set the amplitudes and phases for each harmonic, seeing the resulting waveforms and hearing the sounds. Five waveform presets are included: sine, triangle, square, ramp (sawtooth), and pulse train. The program is free for non-commercial use and can also be downloaded for running offline.

Introduction

For decades, physics teachers have cleverly designed ways for students to visualize finite Fourier series in synthesizing periodic waves, without the need for purchasing an expensive Fourier synthesizer unit. Such methods have used microprocessors [1–3], circuits [4, 5], and computer software [6–9]. These approaches still have the drawback of needing to acquire equipment, purchase software, or install Java. Furthermore, in some cases, there is no sound and not enough synthesizers for each student. The Fourier synthesizer described in this paper solves these problems by providing an HTML5 app that can be easily accessed over the Internet without any plug-ins.

The new Fourier synthesizer incorporates visualization with audio. A main advantage of HTML5 is that no specialized equipment or plug-ins are required and every student gets their own Fourier synthesizer. All one needs is a computer with a modern browser. The program can be used online [10] from the author's website or downloaded if one prefers to run the program offline[1]. The program can also be run using mobile devices such as the iPhone and iPad, either online or offline.

[1] Users with basic programming skills can also modify the program for their own purposes, provided they agree to the non-commercial Creative Commons License included with the computer code.

Amplitude, frequency, and waveform

It is helpful for students to know a little about periodic waves before proceeding. The 'big three' characteristics of a periodic sound wave are amplitude, frequency, and waveform. Amplitude generally correlates with loudness and frequency with pitch. However, we are more sensitive to higher pitches, indicating that higher pitches are perceived to be louder. The waveform, is the shape of the wave on an oscilloscope and provides a visualization of the timbre, or texture of the sound. Students can understand timbre by the following simple example provided by the teacher. Ask the students to consider a violin and flute playing the same pitch at the same loudness. Will they still perceive a difference in the sound? Yes they will say, the flute sounds different than the violin. The feature describing this difference is the waveform, or timbre. Fourier synthesis allows one to construct periodic waves with different timbres and from the construction, the student learns that timbre can be analyzed as a superposition of harmonics.

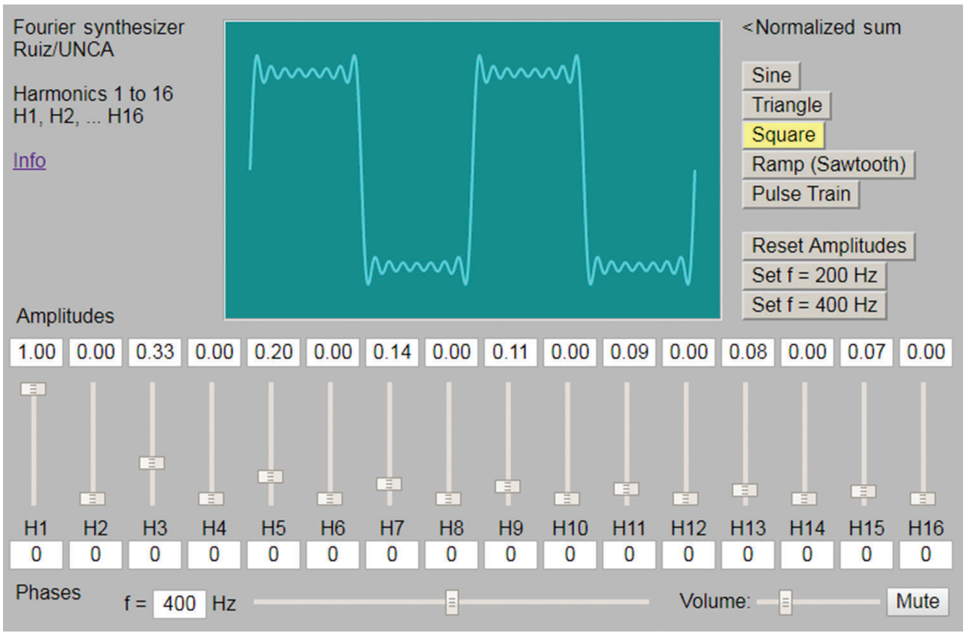

Figure 1. Screenshot of Fourier synthesizer app illustrating the formation of a square wave.

The Fourier synthesizer app

A screenshot of the HTML5 Fourier synthesizer app is provided in figure 1. The amplitudes and phases of the first sixteen harmonics can be set in order to synthesize various periodic waves. Fourier's theorem states that any periodic wave[2] having frequency f can be synthesized from sine waves (harmonics) with frequencies *f*, 2*f*, 3*f*, and so on by choosing their respective amplitudes and phases. The amplitude of each harmonic is a measure of the vertical extent of each sine wave and the phase represents the horizontal positioning of each sine component. Shifting a sine wave 180° to the right means that the wave moves to the right by one-half wavelength. The harmonic components are often referred to as partials.

Students can play with the vertical sliders to smoothly change the individual amplitudes of the harmonics. Values for the amplitudes can also be entered in the text boxes. The phase of each harmonic is entered in the appropriate phase text box below the vertical sliders. The first harmonic, i.e. fundamental frequency, of the periodic wave can be set from 200 Hz to 800 Hz. At the maximum setting, the 16th harmonic has a frequency of 16 (800) = 12 800 Hz. Buttons are included to quickly set the fundamental frequency to 200 Hz, in which case one period of the wave fills the oscilloscope screen. Two full periods are visible when the fundamental frequency is 400 Hz, illustrated in figure 1.

Figure 1 illustrates the synthesis of a square wave. The odd harmonics $m = 1, 3, 5, 7, \ldots 15$ with their respective Fourier amplitudes $1/m$ are included. Note the formation of the 'rabbit ears' at the edges of the periodic wave. Though this overshoot effect is often referred to as the Gibbs phenomenon, it was first discovered by Henry Wilbraham, 'an A.B. of Trinity College, Cambridge, who published an article on this topic in the year 1848'. [12] The mathematical analysis of the overshoot is quite advanced; however, the dramatic visualization of the Gibbs–Wilbraham phenomenon in the Fourier synthesizer allows the non-science student to visualize its formation and appreciate this subtle effect without any math.

The amplitudes and phases for the five presets in the app are given in table 1. The amplitude A_n and phase ϕ_n are defined by $f(t) = A_0 + \sum_{n=1}^{\infty} A_n \sin(n\omega t + \phi_n)$. But I avoid this formula in my liberal-arts *The Physics of Sound and Music* class by labeling the amplitude

[2] Any periodic function satisfying Dirichlet's conditions: 'single-valued and finite' with 'only a finite number of discontinuities and turning-values' in the interval [11].

Table 1. Fourier amplitudes A and phases ϕ in degrees for the five common waveforms.

Waveform	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}	A_{13}	A_{14}	A_{15}	A_{16}
Sine	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Triangle	1	0	1/9	0	1/25	0	1/49	0	1/81	0	1/121	0	1/169	0	1/225	0
Square	1	0	1/3	0	1/5	0	1/7	0	1/9	0	1/11	0	1/13	0	1/15	0
Ramp	1	1/2	1/3	1/4	1/5	1/6	1/7	1/8	1/9	1/10	1/11	1/12	1/13	1/14	1/15	1/16
Pulse train	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Phase	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8	ϕ_9	ϕ_{10}	ϕ_{11}	ϕ_{12}	ϕ_{13}	ϕ_{14}	ϕ_{15}	ϕ_{16}
Sine	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Triangle	0	0	180	0	0	0	180	0	0	0	180	0	0	0	180	0
Square	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Ramp	180	180	180	180	180	180	180	180	180	180	180	180	180	180	180	180
Pulse train	90	90	90	90	90	90	90	90	90	90	90	90	90	90	90	90

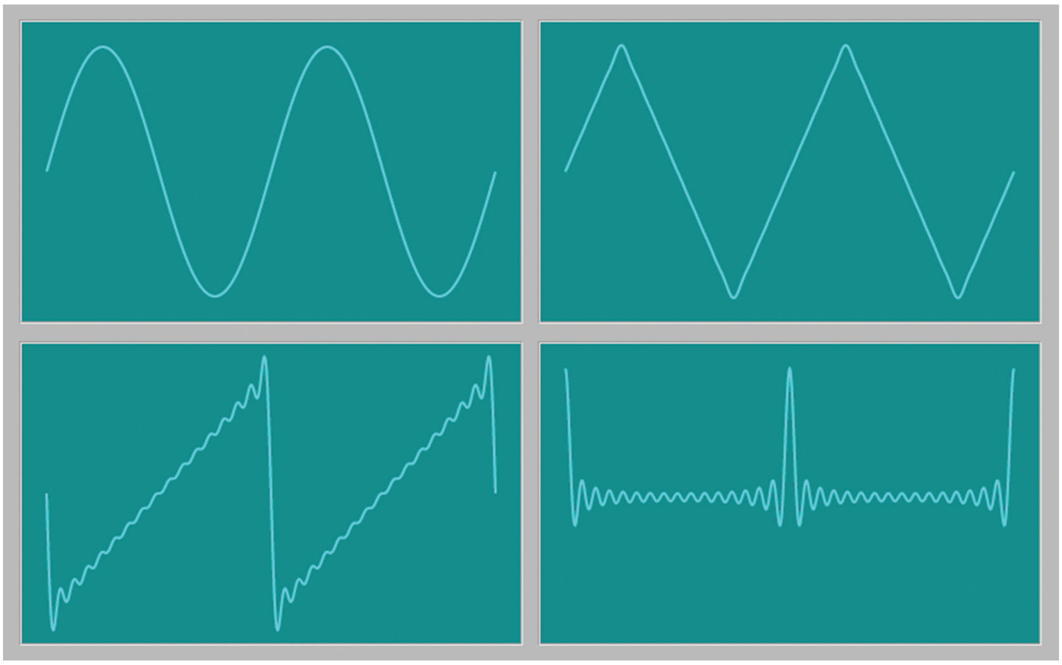

Figure 2. Four synthesized waveforms. The upper two are the sine and triangle; the lower two are the ramp (sawtooth) and pulse train.

for a sine wave I draw on the blackboard and demonstrating phase by sketching a second sine wave shifted by a half wave (a 180° phase shift). The sign convention used in the above formula requires a shift to the left for a positive phase shift.

Note that the ramp waveform in table 1 is often called sawtooth. The waveforms are arranged from top to bottom in order of increasing presence of harmonics for each Fourier spectrum. It is the Fourier spectrum that gives a periodic tone its characteristic timbre or sound quality. The timbre enables one to distinguish between a flute and oboe playing the same pitch at the same loudness.

The simplest Fourier spectrum is that for the sine wave, with only the first harmonic contributing. As the presence of higher harmonics (overtones) increases, the sound becomes denser. The waveform with the richest spectrum is the pulse train, which has raspy timbre. On the other hand, the sine wave timbre, a pure tone, sounds like a whistle.

See figure 1 for the square wave and figure 2 for the four other presets found in table 1. Note the subtle choice of phases required for the triangle wave. Students can investigate what happens when the phases for the triangle are all set to zero or 180°. Similarly, students can explore the effects of changing the phases for the ramp and pulse train waveforms. For example, the ramp wave will slant the other way with phases all set to zero. Interesting variations of the pulse train result when all the phases are set to 0°, then 180°, and finally 270°. Students can be encouraged to discover more waveforms. Playing with the amplitudes and phases provides for a limitless number of timbres.

Math for advanced students

Students in upper-level mathematics and physics classes are typically introduced to Fourier series with sines and cosines. These students might be curious as to why the cosines are missing in the HTML5 app of this paper. The reason is that the app uses the amplitude-phase[3] form for the Fourier series,

$$f(t) = A_0 + \sum_{n=1}^{\infty} A_n \sin(n\omega t + \phi_n), \qquad (1)$$

where $\omega = 2\pi f$.

[3] An alternate expression to equation (1) is the amplitude-phase form with only cosines rather than sines [13].

Equation (1) can be quickly transformed into the common Fourier series with sines and cosines using the trig identity

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta. \quad (2)$$

The result is

$$f(t)=A_0+\sum_{n=1}^{\infty}A_n\left[\sin(n\omega t)\cos(\phi_n)+\cos(n\omega t)\sin(\phi_n)\right]. \quad (3)$$

With the assignments $\frac{a_0}{2}=A_0$, $a_n=A_n\sin\phi_n$, and $b_n=A_n\cos\phi_n$, equation (3) becomes

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos(n\omega t)+b_n\sin(n\omega t)\right], \quad (4)$$

the form most familiar to students in physics and engineering.

Conclusion

The online Fourier synthesizer app introduced in this paper has widespread application for students of all levels. Even students at very young ages who are unfamiliar with physics can play with the controls to synthesize waveforms. In this way such students can be introduced to the physics of timbre through experimental exploration. The visualizations and associated audio bring life to Fourier's theorem for both introductory and advanced students. The professional teacher will also have fun constructing waves and will gain a deeper understanding of Fourier synthesis.

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