Volume 52 Number 2 March 2017 ISSN 0031-9120

Physics*education*

iopscience.org/physed

Tuba physics

Producing 24 harmonics with a fixed pipe length

Papers: Banana STEM, Binding blocks **Frontline:** Pepper's ghost

IOP Publishing

Twenty-Four Tuba Harmonics Using a Single Pipe Length

Bud Holmes[1] and Michael J Ruiz[2]

[1]New Orleans Jazz National Historical Park, New Orleans, LA, USA

[2]Department of Physics, UNC Asheville, Asheville, North Carolina, USA

E-mail: budomu@gmail.com and mjtruiz@gmail.com

Abstract

Harmonics arise naturally from the resonances in strings and pipes. A video demonstration (Ruiz 2016 YouTube: *Tuba Harmonics* https://youtu.be/souhEzOP9c4) is provided where a tubist (coauthor Holmes) produces a phenomenal 24 harmonics using a single tuba pipe length by controlling the buzz of his lips. The frequencies of the harmonics, measured with the free software program Audacity, fall excellently on a linear fit using a spreadsheet. The skillful musical production of so many harmonics with a fixed pipe length is an extraordinary illustration of physics.

The harmonic series

Students are introduced to harmonics in physics when they study standing waves on strings and in pipes. For strings and open pipes, the standing waves form the harmonic series. The frequency of the nth harmonic in the series is given by

$$f_n = n\, f_1 , \qquad (1)$$

with n = 1, 2, 3, …. The frequency f_1 is the frequency of the first harmonic, also called the fundamental. Harmonics where $n \geq 2$ are called overtones, the first overtone being n = 2, the second overtone at n = 3, and so on. For strings and open pipes, the wavelength of the nth harmonic is $\lambda_n = \frac{2L}{n}$, where L is the length of the string or open pipe. Substituting the wavelength into the wave relation $v = \lambda f$, equation (1) can also be written as

$$f_n = \frac{v}{\lambda_n} = \frac{nv}{2L}. \quad (2)$$

For standing waves in closed pipes, $\lambda_m = \frac{4L}{m}$ and $f_m = \frac{mv}{4L}$, where m is odd, i.e. the resonances are odd-numbered harmonics. Note that the fundamental wavelength ($4L$) for a closed pipe of length L is twice the fundamental wavelength ($2L$) for an open pipe of the same length (L). If a student taps an open pipe while another student places a hand over one end, the frequency heard halves since the closed-pipe wavelength is twice that of the open pipe. An open organ pipe can be capped to achieve a lower octave[1] without having to double the length of the open pipe, thus saving vertical space.

[1] Two frequencies form the musical interval of an octave when their frequency ratio is 2:1 or 1:2.

Approximate locations of harmonics on the keyboard

Figure 1 shows the first 16 pitches in the harmonic series starting on C_1 = 32.7 Hz, the lowest C on the piano. The frequency of the lowest note on the piano is A_0 = 27.5 Hz exactly.[2] The frequency for the note one semitone[3] higher can be found by multiplying 27.5 by the 12th root of 2, giving $27.5 \cdot \sqrt[12]{2} = 27.5 \cdot 1.059... = 29.1\,\text{Hz}$. Multiplying by the 12th root of 2 for each rising semitone insures that each higher octave (12 semitones) is exactly double[4] the frequency in equal temperament. An instructive problem is to calculate the frequency of C_1 = 32.7 Hz by multiplying A_0 = 27.5 Hz (the lowest note on the piano) three times with the 12th root of 2.

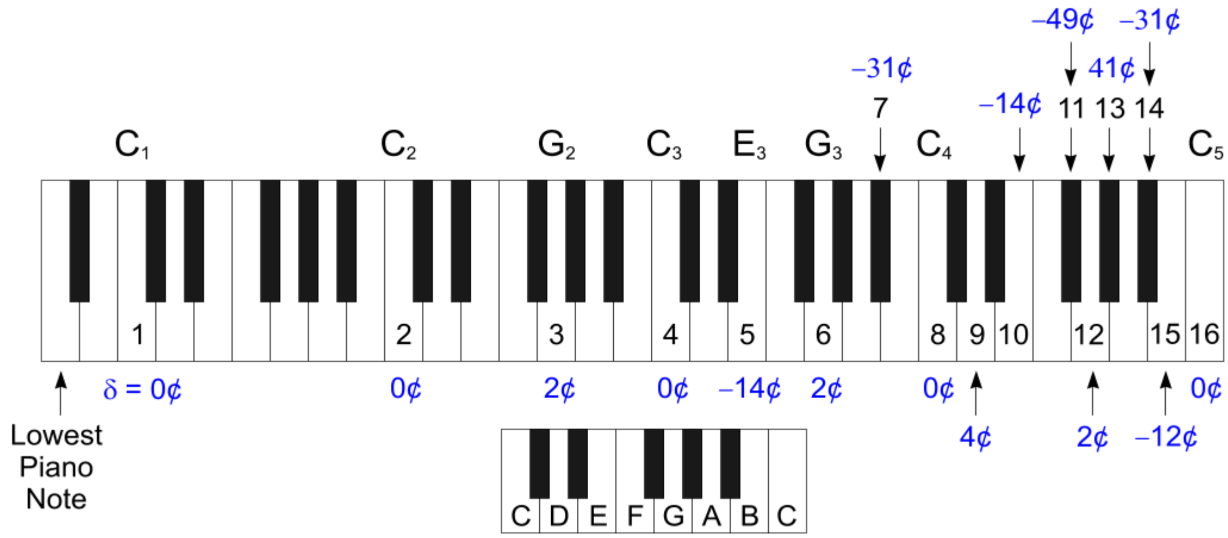


Figure 1. Approximate pitch locations of the first 16 harmonics starting with harmonic 1 on C_1. In each case, the cents (¢) value reveals the deviation from equal-tempered tuning, where a value of 100¢ indicates a semitone.

---

2 Assuming equal-tempered tuning with A_4 = 440.0 Hz (exactly).
3 The interval formed by moving from any note to its adjacent neighbor.
4 Be careful in discussing the piano with your students. The nonlinearity of the stiff piano strings requires tuning each octave slightly higher than a doubled frequency to avoid beats between the first overtone of the lower octave (e.g. C_1) with the fundamental of the higher octave (C_2).

The deviation δ in pitch for each harmonic relative to the nearest matched key in figure 1 is given in cents (¢), where a value of 100¢ equals a semitone. A cent is defined as the 1200th root of 2 so that 1200 cents make up one octave, 100 cents for each of the 12 semitones. The number of cents between the interval defined by a lower frequency f_1 and higher frequency f_2 is given by [2]

$$\text{cents} = 1200 \cdot \log_2 \frac{f_2}{f_1} = 3986 \cdot \log_{10} \frac{f_2}{f_1} \, . \tag{3}$$

As an example for one of the calculations in figure 1, consider the frequency of the sixth harmonic $f_6 = 6 \cdot f_1 = 6 \cdot 32.7\,\text{Hz} = 196.2\,\text{Hz}$ compared to the nearby note $G_3 = 196.0\,\text{Hz}$. Using equation (3), the interval defining the deviation in cents is

$$\delta = 3986 \cdot \log_{10} \frac{f_6}{G_3} = 3986 \cdot \log_{10} \frac{196.2}{196.0} = 1.8 = 2\,¢. \tag{4}$$

Tubas are designed with valve branches so that several harmonic series can be utilized with their associated pipe lengths in order to match the equal-tempered scale. The most refined brass players maintain supple lips with firmly anchored corners so that air speed is the predominant frequency determinant. Breath and facial muscle control produce the lip vibrations that are applied to the mouthpiece. The instrument resonates most when the frequency of lip vibration matches the frequency of the instrument's natural harmonic. Tubists can also tweak individual harmonic resonances shown in figure 1 with deviations of 14 cents or less to bring those pitches in line with equal temperament. However, for our experimental demonstration, the tubist allows each resonance for a fixed pipe length to fall as closely as possible to its natural resonance frequency $f_n = n\, f_1$.

Controlling the general mouth formation (embouchure), lip vibration, lip tension, and the speed of air passing through the lips, the tubist is able to drive the fixed-length pipe into its many resonances. The frequencies of the harmonics obtained in this way are described well by figure 1, which indicates how close each harmonic is to an actual note of the equal-tempered scale.

Basic tuba pipe physics

The tuba is the ideal instrument for obtaining many harmonics because the musician can start with a very low pitch for the first harmonic. The CC Tuba is shown in figure 2 with coauthor and tubist Bud Holmes. The CC Tuba has a minimum length of 16.0 ft (4.88 m). This minimum length is the default length if no valves are pressed. Pressing the valves adds to the default length by including valve branches in the pipe path.

Figure 2. Coauthor and tubist Bud Holmes holding a CC Tuba.

The tuba consists of a mouthpiece (cup and tapered back bore), a mouthpipe (also with a taper), a main conical bore, and a flaring bell. [3] The mouthpiece serves as a closed end and the flaring bell is an open end. An interesting property of a conical pipe forming a complete cone is that all harmonics are present. [4] Consider first a closed cylindrical pipe of length L. The resonance pitches, as discussed earlier, are given by $f_m = \frac{mv}{4L}$, where $m = 1, 3, 5...$, odd numbers.

Now imagine transforming the cylinder into a cone. The resulting frequencies become $f_n = \frac{nv}{2L}$, where $n = 1, 2, 3...$, the harmonics for an open pipe of the same length L. [4] An elegant mathematical derivation of these results along with a beautiful graph showing the transformation of the closed-pipe resonances to the conical case was published by Ayers, Eliason, and Mahgerefteh in 1985 [5].

For interesting discussions of the tuba's cousins, the trumpet and trombone, both of which have considerable lengths of cylindrical tubing [3], see two papers by LoPresto. [6,7] With the trumpet and trombone, the additions of the mouthpiece (with mouthpipe) and bell flare transform the closed-pipe harmonics to almost the same spectrum as an equivalent cone of the same length [8].

Fundamental pitches on the tuba

Fundamentals, also called pedal tones, are accessible on the tuba as true natural harmonics; however, they are not commonly called for in orchestral scores. The tubist at times can encounter the fundamental, e.g. in some 20th century orchestral music. Typically, the tubist plays one of the overtones for each pipe length. [9]

The length of the CC tuba without pressing the valves is 16.0 ft (4.88 m). The corresponding fundamental falls on the note C_1 = 32.7 Hz, illustrated in figure 1 and calculated earlier from A_0 = 27.5 Hz. The effective length L' of the tuba can be inferred by combining $v = \lambda f$ and $\lambda = 2L'$ for the conical pipe to obtain the equation

$$L' = \frac{v}{2f} . \tag{5}$$

Using equation (5) with $v = 345\ \mathrm{m \cdot s^{-1}}$ and $f =\ 32.7\ \mathrm{Hz}$, gives an effective length

$$L' = \frac{345\ \mathrm{m \cdot s^{-1}}}{2 \cdot 32.7\ \mathrm{Hz}} = 5.28\ \mathrm{m}, \tag{6}$$

which is about 8% greater than the physical length $L = 4.88\ \mathrm{m}$. The discrepancy is due to the end correction at the open end and the fact that the tuba is not perfectly conical. A similar situation is encountered with the trombone, where an end correction and flare effect need to be taken into account. [7]

The spectrum of the fundamental played on a CC Tuba is very rich. When the fundamental is played, many harmonics are observed in the frequency spectrum. See figure 3 for the frequencies of some of these harmonics, with H1, H2, H4, H8, and H16 labeled with their corresponding octaves on the keyboard. These frequencies were measured by the free audio software program Audacity. [10] Note that the measured value for the fundamental frequency is 33 Hz, in good agreement with the actual value C_1 = 32.7 Hz.

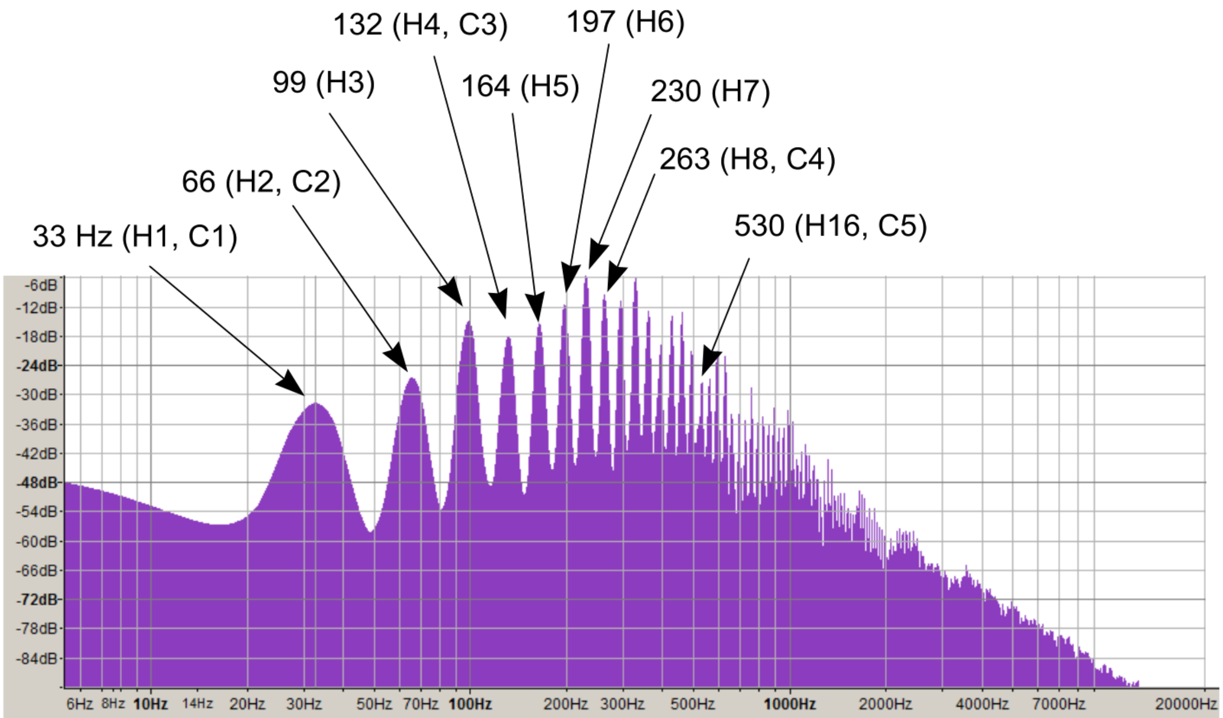


Figure 3. Frequency spectrum in Audacity for the fundamental on the CC tuba playing the lowest C, which is C_1 = 32.7 Hz.

It is interesting to apply a linear best fit for the data in figure 3. The result using an Excel spreadsheet is shown in figure 4. The fit is excellent and supplies an even better experimental value for the fundamental arises, 32.8 Hz as compared with the expected C_1 = 32.7 Hz.

Students can be encouraged to record their own sounds from musical instruments and investigate spectra. The audio source for figure 3 comes from our tuba video [1] (coauthor Holmes) in a large living room (home of coauthor Ruiz). A video editing program was used to separate the audio from the video for spectral analysis with the free program Audacity [10] to produce figure 3. Alternatively, the built-in microphone of a portable computer can be used to record sounds with available software such as Apple's GarageBand [11]. There are also apps that allow students to record and analyze sounds using their smartphones and tablet PCs [12].

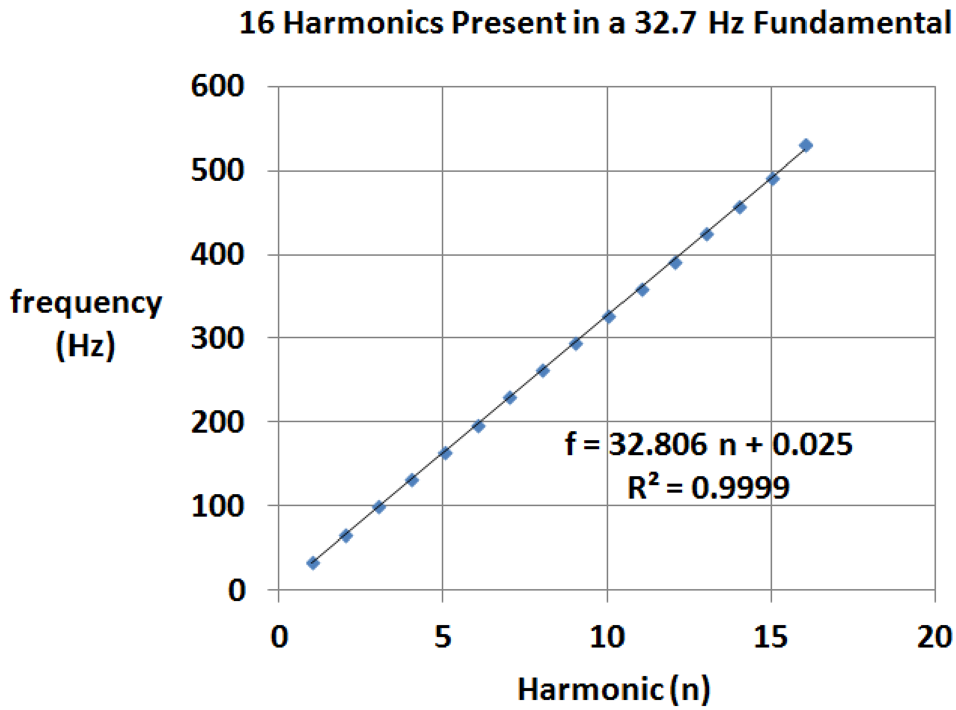


Figure 4. Best fit for the first 16 harmonics in the spectrum of the 32.7 Hz CC tuba fundamental, giving an experimental value of 32.8 Hz for the fundamental.

Playing 16 harmonics on a CC tuba with no valves pressed

When no valves are pressed, the tuba's length is given by the main bugle section without adding any of the valve branches. For the CC tuba this length is 16.0 ft (4.88 m). The tubist in the video of reference 1, with skillful buzzing of the lips, plays 16 harmonics with this fixed main bugle section of the tuba. Therefore, the fundamental starts on C_1 = 32.7 Hz.

Since the harmonics in general do not fall perfectly on keyboard frequencies as discussed earlier and shown in figure 1, tubists use valve branches and embouchure to match equal-tempered tones. The tubist in the video did not try to obtain equal-tempered tones. Instead, the tubist aimed for the natural harmonic resonance of the tuba in each case. The tubist quickly found the sweet spot, i.e. where the tuba responded with its most sympathetic vibration, for each harmonic. Theory predicts that these natural harmonics follow $f_n = n\, f_1$ of equation (1).

After quickly finding a natural harmonic, the tubist sustained the pitch as steady as possible. He then went on to promptly locate the next harmonic and so on. The result is plotted in figure 5 where the best fit is made. Note the nice correlation with the theoretical equation $f_n = 32.7\,n$ for the minimum length of 16.0 ft (4.88 m) of the CC tuba.

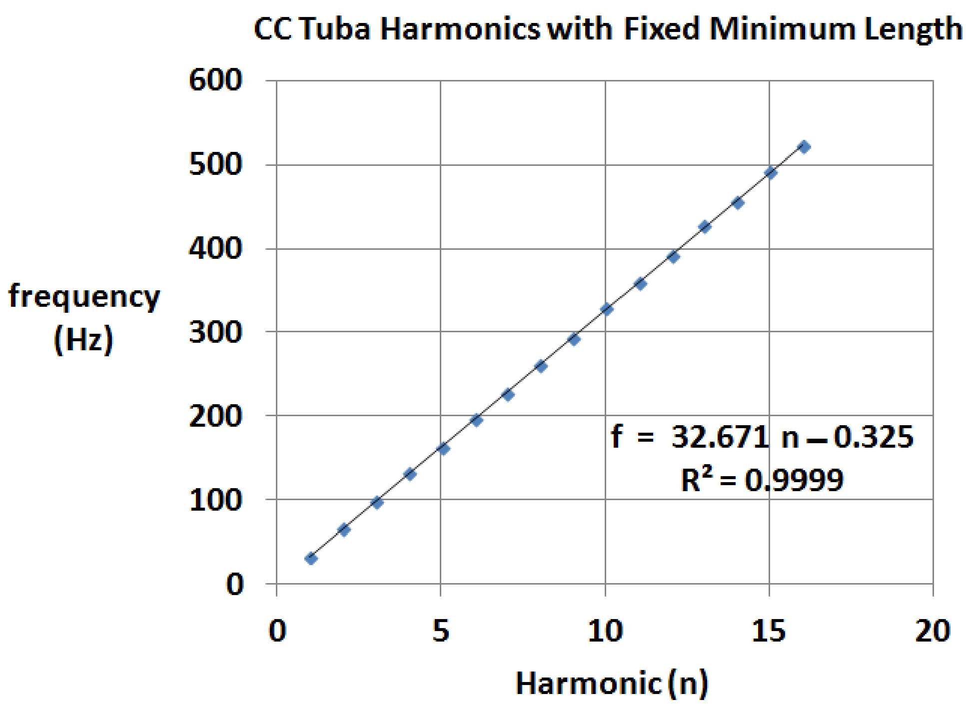


Figure 5. Best fit for the first 16 harmonics individually played on the CC tuba with the fixed minimum bugle length of 4.88 m, which corresponds to a fundamental of C_1 = 32.7 Hz.

Playing 24 harmonics on the F tuba with all 5 valves pressed

The F tuba, shown in figure 6, has a base length equal to 12.0 ft (3.66 m). The theoretical fundamental frequency for this base length is predicted from equation (2),

$$f_1 = \frac{v}{2L} = \frac{345\ \mathrm{m \cdot s^{-1}}}{2 \cdot 3.66\ \mathrm{m}} = 47.13\ \mathrm{Hz}\,. \qquad (7)$$

But a discrepancy is expected since the tuba is not an ideal cone and there is the end correction as discussed earlier. The base length of the F tuba is designed to obtain the pitch F_1 = 43.65 Hz.

Checking the discrepancy between 43.65 Hz and 47.13 Hz, it is found to be 8% as noted earlier with the CC tuba.

Figure 6. Coauthor and tubist Bud Holmes holding an F tuba and pressing all five values to obtain the maximum tuba pipe length.

An instructive calculation for the student is to determine the keyboard frequency F_1 = 43.65 Hz from the nearest A. Since A_4 = 440 Hz, one can easily keep dividing by 2 (dropping octaves) to reach A_1 = 55 Hz. Then from the letter designations in figure 1, it is apparent that one needs to divide by the 12th root of 2 a total of four times to drop down the 4 semitones to arrive at F_1 = 43.65 Hz.

To obtain the largest number of harmonics on the tuba it is best to start with as low a frequency as possible. The longest pipe length is desired. When all five valves are pressed, all valve pipe sections are added to the base length. The five valve branches take the pitch down 11 semitones, almost an octave. The result is the incredibly low $F\#_0$, three semitones below the lowest note on the piano. A quick way to determine the frequency of $F\#_0$ is to first drop down an octave from F_1 = 43.65 Hz to F_0 = 43.65 Hz / 2 = 21.83 Hz. Then move up a semitone to arrive at $F\#_0$ = 21.83 · 1.059 = 23.1 Hz. Starting with the lowest possible pitch gives the tubist much frequency space to move up the harmonic series achieving the most resonances.

The tubist in our video abstract [1] attains 24 harmonics for $F\#_0$ = 23.1 Hz. The first 20 of these harmonics are fitted with the best linear fit in figure 7. The last four harmonics, shown in red, are not used since these frequencies are extremely difficult to hit, thus introducing considerable uncertainty. Note how the last four harmonics drift slightly upward and away from the best fit. In musical terms, these pitches are getting sharper and sharper. The equation for the fit, taking n = 1, gives a measured fundamental $f_1 = 23.841 - 0.8263 = 23.0$ Hz for the $F\#_0$ = 23.1 Hz note.

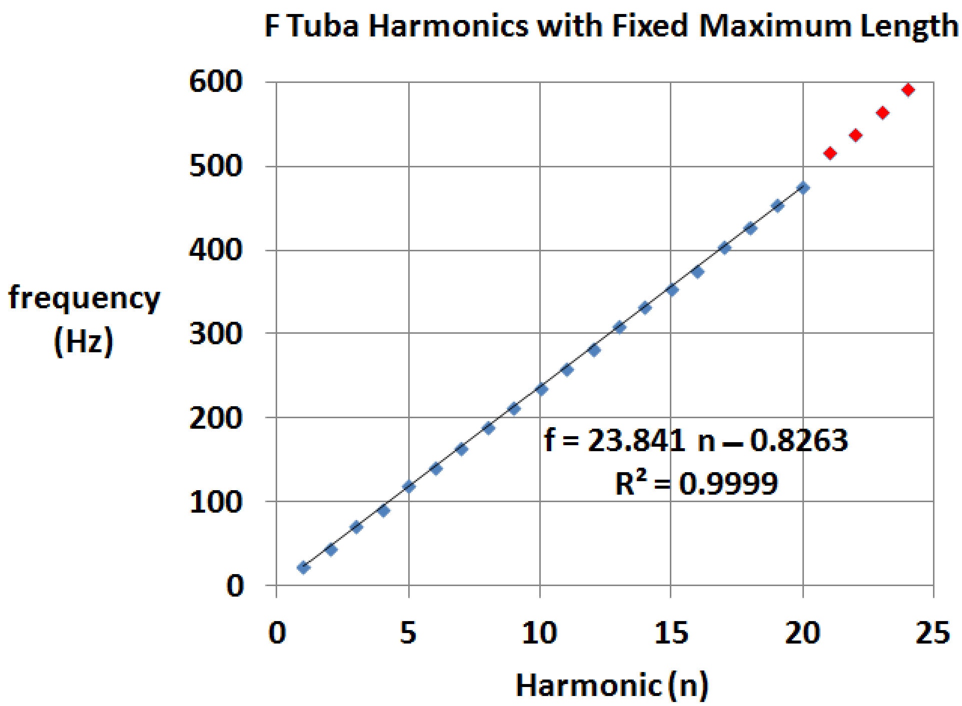


Figure 7. Best fit for the first 20 harmonics individually played on the F tuba with its fixed maximum pipe length, corresponding to a fundament $F\#_0$ = 23.1 Hz. Harmonics 21 to 24 (the last four data points drifting upward) were not included in the fit due to the extreme difficulty in achieving these.

The CC tuba could have been used with pressed valves to produce more than the 16 harmonics shown earlier in figure 5. However, the F tuba was introduced because the F tuba is smaller in length. The shorter minimum bugle section is 12 ft (3.66 m) rather than the 16 ft (4.88 m) of the CC tuba. Less energy is required on the part of the tubist to drive the shorter system into its resonances. Using less energy, the tubist can more precisely sustain the harmonics. Therefore, the many harmonics played sound cleaner, resulting in better recorded data for physics analysis.

Conclusion

The harmonic series naturally appears when teachers introduce students to waves on strings and in pipes. The formula for the harmonic series $f_n = n\, f_1$, equation (1), is all the student needs to know in order to appreciate the basic physics of the tuba video of harmonics. [1] Before showing the video, the teacher can ask students how many harmonics do they think can be performed on a bugle or pipe of fixed length? The discussion can begin with songs composed using harmonics, which songs were the only ones bugle players could play before the arrival of valves (or the slide of a trombone). As an example, the melody *Taps* employs only four harmonics, namely, harmonics H3, H4, H5, and H6.

The first 4 notes in the opening theme to *Also sprach Zarathustra* by Richard Strauss (1896) dramatically uses harmonics H2, H3, H4, and H5. Students will most likely know this theme from Stanley Kubrick's movie *2001: A Space Odyssey* (1968), screenplay by Kubrick and science fiction author Arthur C. Clarke. The first four notes of the theme can be achieved by whirling the flexible toy corrugated tube. [13] Energetic students might go beyond to obtain 5 or 6 harmonics twirling the toy tube. After this initial discussion, asking students again about a realistic number of harmonics achievable on a pipe with fixed length will most likely lead to estimates lower than 10.

Students will not imagine that 16 harmonics can be produced with a pipe of fixed length. The performance of 24 harmonics on the F tuba will astound them. To enable the teacher to build the suspense, we have prepared video excerpts [14, 15] from our video abstract [1] so that your students can count the harmonics without knowing in advance how many harmonics will be obtained with the CC tuba [14] and the F tuba [15].

The tuba is such a cool instrument for achieving many harmonics because the starting pitch is so low. With the F tuba, a tubist can begin with the incredibly low frequency of 24 Hz, lower than the lowest note on the piano. Hearing a fundamental pitch near the lower threshold of human hearing is in itself an interesting physics demonstration. The topic of tuba harmonics is rich in physics, the mathematics of logarithms, and musical tones. Such an interdisciplinary demonstration will appeal to many students.

References

[1] Ruiz M J 2016 YouTube: *Tuba harmonics* https://youtu.be/souhEzOP9c4

[2] Howard D M and Angus J 2006 *Acoustics and psychoacoustics* 3rd edn (London: Elsevier) pp 389-90

[3] Benade A H 1973 The physics of brasses *Sci. Am.* **229** (1) 24-35

[4] Backus J 1977 *The acoustical foundations of music* 2nd edn (London: W. W. Norton & Company) pp 70-1

[5] Ayers R. D., Eliason L J, and Mahgerefteh D 1985 The conical bore in musical acoustics *Am. J. Phys.* **53** (6) 528-37

[6] LoPresto M C 2003 Experimenting with brass musical instruments *Phys. Educ.* **38** (4) 300-8

[7] LoPresto M C 2014 Mouthpiece and bell effects on trombone resonance *Phys. Teach.* **52** (1) 26-9

[8] Campbell M and Greated C 1987 *The musician's guide to acoustics* (London: J M Dent & Sons Ltd) p 333-7

[9] Vienna Symphonic Library 2016 Bass tuba sound production https://vsl.co.at/en/Bass_tuba/Sound_Production (Accessed 4 January 2017)

[10] Audacity at sourceforge 2016 https://sourceforge.net/projects/audacity or the Audacity website at www.audacityteam.org (Accessed 4 January 2017)

[11] Keeports D 2010 Demonstrating sound with music production software *Phys. Educ.* **45** (3) 243-8

[12] Klein P, Hirth M, Gröber S, Kuhn J, and Müller A 2014 Classical experiments revisited: smartphones and tablet PCs as experimental tools in acoustics and optics *Phys. Educ.* **49** (4) 412-8

[13] Ruiz M J 2016 The "2001" theme on the whistling tube *Little Gems* Column Editor Chiaverina C *Phys. Teach.* **54**, 310-1

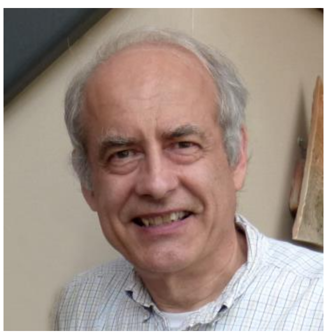
[14] Ruiz M J 2016 *YouTube: CC Tuba harmonics* CC https://youtu.be/uSCxtFUgmmQ so that students can count the harmonics

[15] Ruiz M J 2016 *YouTube: F Tuba harmonics* https://youtu.be/98COh4Hv1XU so that students can count the harmonics

Authors

Bud Holmes is Interpretation Ranger at New Orleans Jazz National Historical Park. He received

his MM in tuba performance from Carnegie Mellon University, USA. Holmes previously served as Principal Tuba for the Asheville Symphony Orchestra and the United States Air Force Heritage of America Band. He uses the harmonic series and seven valve combinations to perform jazz programs at the Old US Mint in the French Quarter.



Michael J Ruiz is professor of physics at the University of North Carolina at Asheville (UNC Asheville), USA. He received his PhD in theoretical physics from the University of Maryland, USA. His innovative courses with a strong online component aimed at general students have been featured on CNN.