Physics I with Calculus, Prof. Ruiz (Doc), UNC-Asheville (1978-2021), [DoctorPhys on YouTube]()
Chapter N. Angular Momentum. **Prerequisite: Calculus I. Corequisite: Calculus II.**

N0. Conservation of Angular Momentum. Below we list translational and rotational formulas. We have seen most of these formulas in earlier chapters.

Translational Physical Quantities				Rotational Physical Quantities			
Quantity	Sym-bol	Definition	Units	Quantity	Sym-bol	Definition	Units
Position	x	Measure from a reference.	m	Angle	θ	Measure from a reference.	rad
Velocity	v	$\frac{dx}{dt} = \lim_{t \to 0} \frac{\Delta x}{\Delta t}$	$\frac{\mathrm{m}}{\mathrm{s}}$	Angular Velocity	ω	$\frac{d\theta}{dt} = \lim_{t \to 0} \frac{\Delta \theta}{\Delta t}$	$\frac{\mathrm{rad}}{\mathrm{s}}$
Acceleration	a	$\frac{dv}{dt} = \lim_{t \to 0} \frac{\Delta v}{\Delta t}$	$\frac{\mathrm{m}}{\mathrm{s}^2}$	Angular Acceleration	α	$\frac{d\omega}{dt} = \lim_{t \to 0} \frac{\Delta \omega}{\Delta t}$	$\frac{\mathrm{rad}}{\mathrm{s}^2}$
Mass	m	With Force. See Below.	kg	Moment of Inertia	I	$I = \sum_i m_i r_i^2$	$\mathrm{kg} \cdot \mathrm{m}^2$
Momentum	p	$p = mv$	$\mathrm{kg} \cdot \frac{\mathrm{m}}{\mathrm{s}}$	Angular Momentum	L	$L = I\omega$	$\frac{\mathrm{kg} \cdot \mathrm{m}^2}{\mathrm{s}^2}$
Force	F	$F = ma$	N	Torque	τ	$\tau = I\alpha$	$\mathrm{N} \cdot \mathrm{m}$
		$F = \frac{dp}{dt}$	N			$\tau = \frac{dL}{dt}$	$\mathrm{N} \cdot \mathrm{m}$
Work	W	$W = \int F dx$	$\mathrm{N} \cdot \mathrm{m}$ or J	Work	W	$W = \int \tau d\theta$	$\mathrm{N} \cdot \mathrm{m}$ or J
Translational Kinetic Energy	K	$K = \frac{1}{2} mv^2$	$\mathrm{N} \cdot \mathrm{m}$ or J	Rotational Kinetic Energy	K	$K = \frac{1}{2} I\omega^2$	$\mathrm{N} \cdot \mathrm{m}$ or J
Impulse	J	$J = \int F dt$ $J = \Delta p$	$\mathrm{N} \cdot \mathrm{s}$	Angular Impulse	J_{Rot}	$J_{\mathrm{Rot}} = \int \tau dt$ $J_{\mathrm{Rot}} = \Delta L$	$\mathrm{N} \cdot \mathrm{m} \cdot \mathrm{s}$ or $\mathrm{J} \cdot \mathrm{s}$

Connecting Equations

$$x = \theta r \qquad v = \omega r \qquad a = \alpha r$$

$$\vec{\tau} = \vec{r} \times \vec{F} \qquad \vec{L} = \vec{r} \times \vec{p}$$

Note that translational impulse and rotational impulse have different units. Remember that with impulse, the time interval is very short so we could write

$$J = F\Delta t = \Delta p \quad \text{and} \quad J_{\text{Rot}} = \tau \Delta t = \Delta L \, .$$

When there are no external forces, $F = 0$ and $F = \dfrac{dp}{dt} = 0$ leads to $p = const$. Similarly, if there are no external torques, $\tau = 0$ and $\tau = \dfrac{dL}{dt} = 0$ leads to $L = const$. When you consider the universe as a whole, there are no external anything because the universe includes everything. So we can list two general conservation laws as:

conservation of momentum,

conservation of angular momentum.

And we do not want to forget our other law:

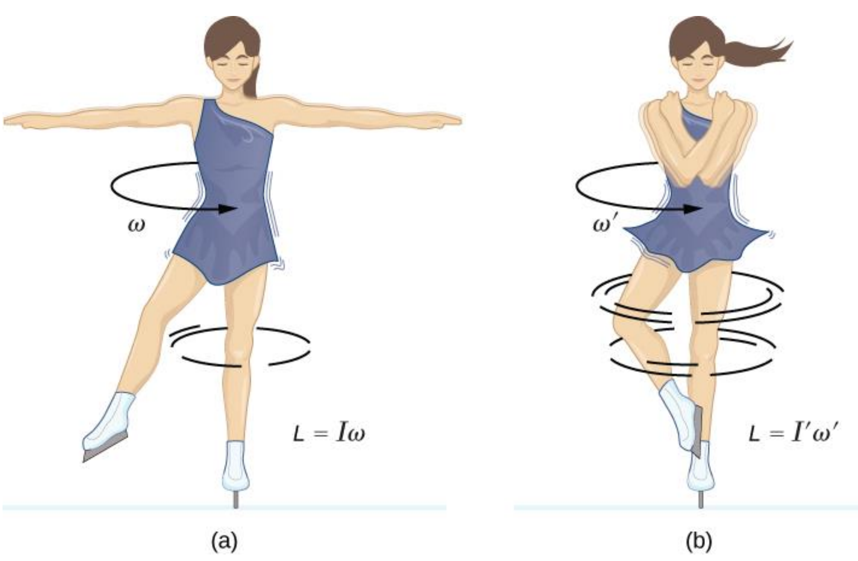
conservation of energy,

where energy includes energy in the form of matter.

We proceed now to problems and solutions.

N1. The Spinning Figure Skater.

Image Courtesy Rice University. License: Creative Commons Attribution 4.0
Text by Charles Niederriter. Available for free at openstax



We can use conservation of angular momentum to understand how an ice skater spins faster when the arms are pulled in.

There are no external forces once the ice skater gets spin in figure (a). The angular momentum is given by the formula

$$L = I\omega \, .$$

When the ice skater makes more of her mass hug the vertical axis of rotation, the new moment of inertia I' decreases compared to the initial I. Therefore, $I' < I$. But since angular momentum is conserved, we have

$$I\omega = I'\omega'.$$

The decrease in rotational inertia, i.e., $I' < I$, means an increase in angular velocity to keep the above equation true. The result is faster spinning,

$$\omega' > \omega.$$

Courtesy deerstop, Wikimedia, Dedicated to the Public Domain.

"Yuko Kawaguti, in the 2010 Cup of Russia, free skating. This photo is an example of a classical illustration of conservation of angular momentum in physics. When a spinning figure skater pulls in her arms, reducing her moment of inertia, she rotates faster." Photographer deerstop, Wikipedia

Problem 1. A skate starts a spin with an arm and leg extended with a frequency of 1 rotation per second, which we express as $f_1 = 1\text{ Hz}$, where Hz = hertz stands for 1 per second. Rotation is understood. Her initial moment of inertia is I_1. She then pulls in her arms and legs as close to the axis of rotation as possible to achieve a moment of inertia I_2 with a corresponding rotational frequency of $f_2 = 5\text{ Hz}$. Express I_2 in terms of I_1.

Since there are no external torques affecting the spin, we can invoke conservation of angular momentum $L = I\omega$.

$$L_1 = L_2 \quad \Rightarrow \quad I_1\omega_1 = I_2\omega_2$$

Note that $\omega = 2\pi f$. Therefore $I_1\omega_1 = I_2\omega_2$ leads to $I_1 2\pi f_1 = I_2 2\pi f_2$.

$$I_1 f_1 = I_2 f_2 \quad \Rightarrow \quad I_2 = I_1\frac{f_1}{f_2} \quad \Rightarrow \quad I_2 = I_1\frac{1\text{ Hz}}{5\text{ Hz}} \quad \Rightarrow \quad I_2 = \frac{1}{5}I_1$$

By the way, spinning at 5 Hz is equivalent to 5 x 60 = 300 rotations per minute, often written as 300 rpm. But note that the spinning only lasts a few seconds at this high rate. An 11 year-old young lady, Olivia Rybicka-Oliver from Nova Scotia reached a spin rate of 342 rpm in Warsaw, Poland in 2015. That rate is an incredible

$$f = 342\frac{1}{\text{min}} = 342\frac{1}{\text{min}}\frac{1\text{min}}{60\text{ s}} = \frac{342}{60}\frac{1}{\text{s}} = 5.7\text{ Hz}.$$

Courtesy deerstop, Wikimedia, Dedicated to the Public Domain.
Skater Yuko Kawaguti

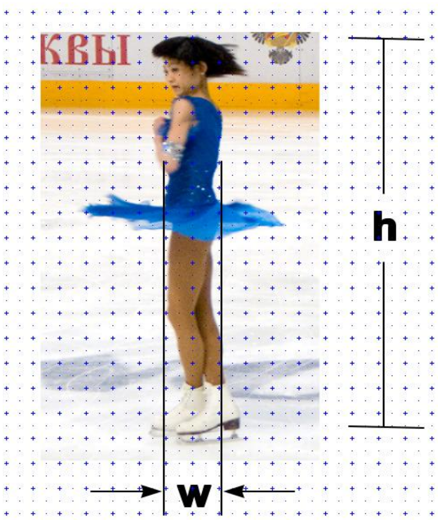
Problem 2. A 36-kg (80 lb) skater spins at 300 rpm when her arms are pulled in. The skater is 152 cm tall (5 feet).

Approximate the spinning skater as a vertical spinning rod. Using this approximation model for the skater, calculate

(i) the moment of inertia of the skater,
(ii) the angular momentum of the skater.

Hint: Use the photo at the left to help you estimate the radius to use in your model. The skater in the photo is close to 152 cm tall.

Solution. (i) Moment of Inertia. The moment of inertia for a cylinder is $I = \frac{1}{2}MR^2$, where M is the mass of the cylinder and R is the radius. Let the distance from each dot to the next be x.



Then, the height $h = 15.5x$ and the width in the figure is $w = 2x$. Since we are given $h = 152 \text{ cm}$, we find

$$w = h\frac{2x}{15.5x} = \frac{2}{15.5}h = \frac{2}{15.5}152 = 19.6 = 20 \text{ cm}$$

Such a detailed analysis is NOT needed for an estimate. You can eyeball the width and even make it greater since the front view shoulder-to-shoulder width would be greater.

The radius is $r = \frac{w}{2} = 10 \text{ cm}$.

$$I = \frac{1}{2}MR^2 = \frac{1}{2}36 \text{ kg}\left(0.1 \text{ m}\right)^2 = \frac{1}{2}\frac{36}{100}\text{kg}\cdot\text{m}^2 = 0.180 \text{ kg}\cdot\text{m}^2 \quad \Rightarrow \quad \boxed{I = 0.18 \text{ kg}\cdot\text{m}^2}$$

(ii) Angular Momentum. Use the angular momentum formula $L = I\omega$ with

$$I = 0.18 \text{ kg}\cdot\text{m}^2 \text{ and } \omega = 2\pi f = 2\pi\cdot(1\frac{1}{\text{s}}) = 2\pi\frac{1}{\text{s}}$$

$$L = I\omega = (0.180 \text{ kg}\cdot\text{m}^2)(2\pi\frac{1}{\text{s}})$$

Since we are estimating, just report one significant figure.

$$\boxed{L = 1\,\text{kg}\cdot\frac{\text{m}^2}{\text{s}}}$$

The units $\text{kg}\cdot\frac{\text{m}^2}{\text{s}} = \text{kg}\cdot\frac{\text{m}}{\text{s}^2}\cdot\text{m}\cdot\text{s} = \text{N}\cdot\text{m}\cdot\text{s} = \text{J}\cdot\text{s}$. So an alternative answer is in joule seconds.

$$\boxed{L = 1\,\text{J}\cdot\text{s}}$$

N2. Jumping on a Playground Merry-Go-Round.

Merry-Go-Round Photo Courtesy J, flickr, AKA Yuek Hahn.
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Problem. Running and Jumping On. For the merry-go-round at the left, we will neglect the mass of the rails compared to the solid thick metal disk platform.

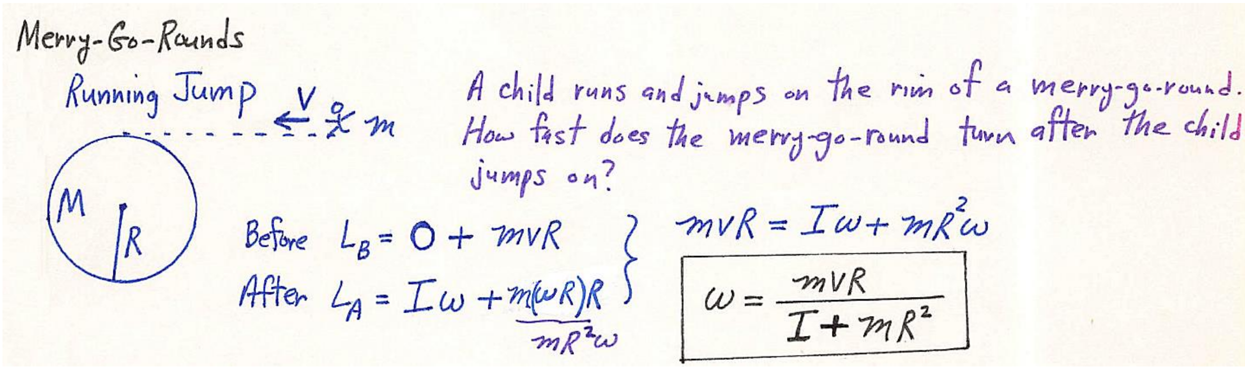
Let the mass of the disk be M and the radius R. A person of mass m runs to the stationary merry-go-round with speed v tangent to the rim and jumps on.

(i) What is the angular velocity of the merry-go-round in terms of M, R, m, and v?

(ii) What is the angular velocity if M = 450. kg (a weight of 1000 lb), R = 1.5 m (5 ft), m = 50. kg (a weight of 110 lb), and v = 3.0 m/s (7 miles per hour)?

(iii) How long does it take the merry-go-round to make one rotation if you neglect friction?

Below is a figure from my old notes of many decades ago sketching out this kind of problem. The equations for the conservation of angular momentum are given with the basic solution.



(i) General Formula. The angular momentum before is tricky. We have the person running with speed v tangent to the top rim of the disk. Right at the rim just before the person jumps on, the angular momentum is

$$L = rp = Rmv\,,$$

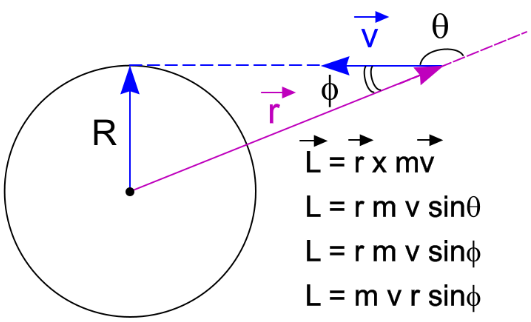
which I like to write as

$$L = mvR\,.$$

Here are a couple of important points:

(1) We need to choose a reference from which angular momentum is calculated. The obvious choice here is the center of the merry-go-round. That choice is the one we implicitly made above in calculating the angular momentum for the incoming running child.

(2) We can calculate the angular momentum due to the running person relative to this center using the vector formula. The figure below shows that the initial angular momentum does not change as the child approaches the merry-go-round.



We start with the formula

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}\,.$$

Working through the cross product, we obtain

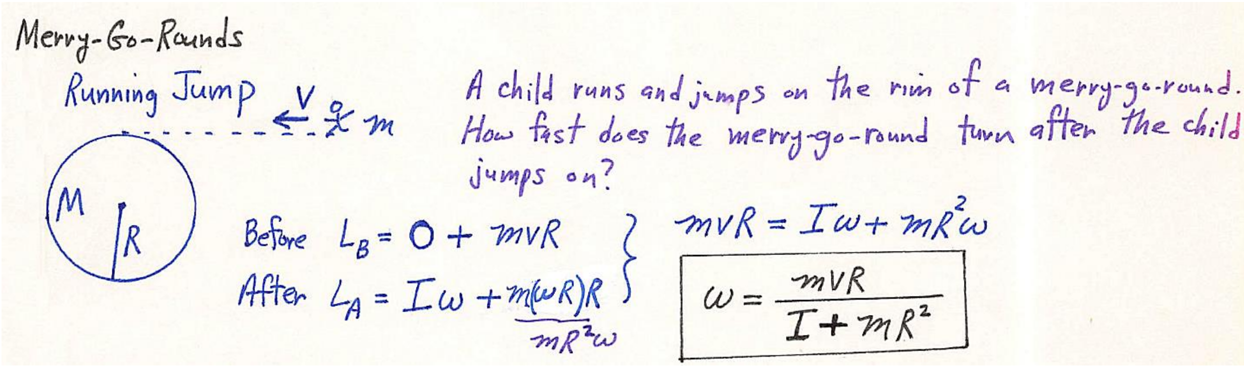
$$\vec{L} = rmv \sin\theta n\,,$$

where the unit vector n points out of the page.

Since the sines of supplementary angles are equal, we can write for the magnitude of the angular momentum

$$L = rmv \sin\theta = rmv \sin\phi = mvr \sin\phi = mvR\,.$$

Returning to our figure with the calculation, we can list the before angular momentum as zero for the stationary disk platform plus our mvR.



$$L_{before} = 0 + mvR$$

Afterwards, the disk starts spinning with the child on the end.

$$L_{after} = I_{disk}\omega + I_{child}\omega$$

Leave the moment of inertia of the disk as I_{disk} for now. The moment of inertia for the child is $I_{child} = mR^2$. The angular velocity is the same from both the disk and child as they will make one revolution in the same amount of time. Conservation of angular momentum can be written as

$$L_{before} = L_{after} \quad => \quad 0 + mvR = I_{disk}\omega + I_{child}\omega \quad => \quad mvR = (I_{disk} + I_{child})\omega$$

$$\omega = \frac{mvR}{I_{disk} + I_{child}} \quad => \quad \omega = \frac{mvR}{I_{disk} + mR^2} \quad => \quad \boxed{\omega = \frac{mvR}{I + mR^2}}$$

The above formula allows the railings to be included. But since we can neglect these,

$$\omega = \frac{mvR}{I + mR^2} \quad => \quad \omega = \frac{mvR}{I_{disk} + mR^2} \quad => \quad \omega = \frac{mvR}{\frac{1}{2}MR^2 + mR^2}$$

$$\omega = \frac{2mvR}{MR^2 + 2mR^2} \quad => \quad \omega = \frac{2mv}{MR + 2mR} \quad => \quad \boxed{\omega = (\frac{2m}{M + 2m})\frac{v}{R}}$$

Is the answer reasonable? The units check out. What about a super massive platform?

$$\lim_{M \to big} \omega = \lim_{M \to big} (\frac{2m}{M + 2m})\frac{v}{R} \approx 0$$

This result is what we would expect.

(ii) Specific Case. We are given M = 450. kg, R = 1.5 m, m = 50. kg, and v = 3.0 m/s.

$$\omega = (\frac{2m}{M+2m})\frac{v}{R} = (\frac{2\cdot 50}{450+2\cdot 50})\frac{3}{1.5} = \frac{100}{550}\cdot 2 = \frac{10}{55}\cdot 2 = \frac{2}{11}\cdot 2 = \frac{4}{11} = 0.3636\,\frac{\text{rad}}{\text{s}}$$

$$\boxed{\omega = 0.36\,\frac{\text{rad}}{\text{s}}}$$

(iii) Time to Make One Rotation (neglecting friction). Friction of course will kick in here but if the merry-go-round is lubricated well, you will not get much slowing down during the first single rotation.

One rotation corresponds to 2π radians. We can quickly arrive at the answer from the dimensions. The time for one rotation, the period T, is

$$T = \frac{2\pi}{\omega} = \frac{2\pi\text{ rad}}{0.3636\,\frac{\text{rad}}{\text{s}}} = \frac{2\pi}{0.3636}\text{s} = 17.28\text{ s}$$

$$\boxed{T = 17\text{ s}}$$

Two formulas relevant here that are important to know by heart are

$$\omega = 2\pi f \text{ and } f = \frac{1}{T}.$$

The first equation we have seen in our study of rotation. The second one can be thought of as common sense. If it takes T = 1/2 second to do something one, then you are doing f = 2 per second. You flip both the number and the unit: T = 0.5 s and f = 1/T = 1/(0.5 s) = 2 1/s = 2 hertz = 2 Hz. Same goes the other way. If you do something 2 times per second, it takes 1/2 second to do it once.

Approaching the problem this way with these formulas:

$$\omega = 2\pi f \quad \Rightarrow \quad \omega = 2\pi\frac{1}{T} \quad \Rightarrow \quad T = \frac{2\pi}{\omega} \quad \Rightarrow \quad T = \frac{2\pi}{0.3636} = 17\text{ s}.$$

As we have said, it is always good to know more than one way to do things.

N3. Running in a Circle on a Playground Merry-Go-Round. This problem is a little far-fetched since first, it is hard to come by a playground merry-go-round without all the railings. Second, it is hard to walk, let along run, on a rotating platform.

Merry-Go-Round with No Handles, Halle Playground, Berlin, Germany (January 1, 2003)

Courtesy Andrew Coffin, flickr, License: [Attribution-NonCommercial-NoDerivs 2.0]

We found a merry-go-round with no handles. Imagine a large rotating platform and level.

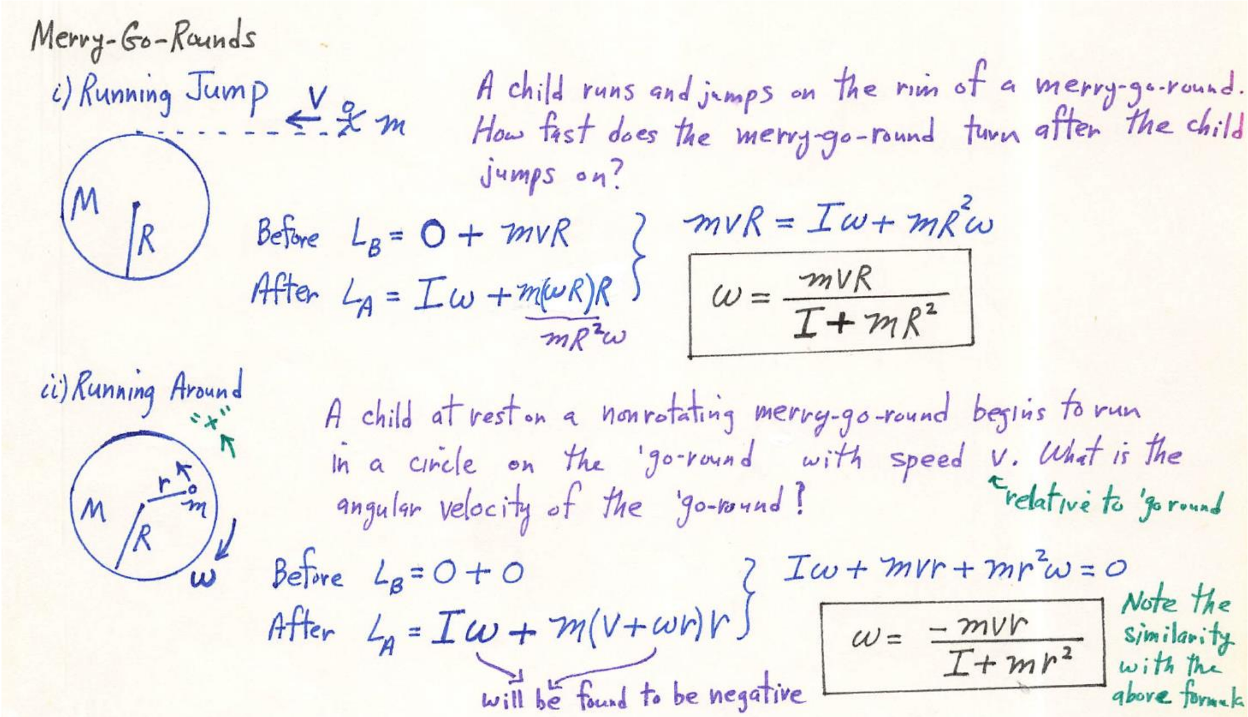
Problem. A child is at rest on a large merry-go-round with no rails. The merry-go-round is stationary at the start and have a moment of inertia I. Then the child of mass m stands at a distance r from the center to walk or run in a circle at that radius r on the 'go-round with speed v relative to the platform.

(i) What is the angular velocity of the 'go-round in terms of the relevant parameters? We do not expect the child to walk for any long length of time.

(ii) What is the formula if the platform is a disk with mass M and radius R?

(iii) What is the angular velocity if M = 450. kg (a weight of 1000 lb), R = 1.5 m (5 ft), m = 50. kg (a weight of 110 lb), v = 0.50 m/s (1 mile per hour) and r = 1.2 m (4 ft)?
(iv) How long does it take to make one rotation neglecting friction?

A sketch and solution with conservation of angular momentum is in the figure part ii below.



Before the child moves, the total angular momentum is zero since the angular momentum for the child and the angular momentum for the disk are both equal to zero. Conservation of angular momentum then gives

$$0 + 0 = I_{disk}\omega_{disk} + I_{child}\omega_{child} \quad => \quad I_{disk}\omega_{disk} + I_{child}\omega_{child} = 0$$

The easier parameters are these three.

$$I_{disk} = \frac{1}{2}MR^2 \qquad \omega_{disk} = \omega \qquad I_{child} = mr^2$$

The tricky one is the angular velocity of the child since the child is moving on the disk.

$$\omega_{child} = \frac{v}{r} + \omega$$

Think of this equation as the result relative to the ground equals the result relative to the disk plus the disk's result relative to the ground. Putting it all together,

$$I_{disk}\omega_{disk} + I_{child}\omega_{child} = 0 \quad \text{and} \quad I_{disk}\omega + mr^2(\frac{v}{r} + \omega) = 0,$$

where I intentionally left the disk inertia as a variable to compare with the above figure.

$$I_{disk}\omega + mrv + mr^2\omega = 0$$

$$(I_{disk} + mr^2)\omega + mvr = 0$$

$$(I_{disk} + mr^2)\omega = -mvr$$

$$\omega = -\frac{mvr}{I_{disk} + mr^2}$$

This answer agrees with the calculation in the figure.

Now, inserting $I_{disk} = \frac{1}{2}MR^2$,

$$\omega = -\frac{mvr}{\frac{1}{2}MR^2 + mr^2}$$

$$\boxed{\omega = -\frac{2mvr}{MR^2 + 2mr^2}}$$

Note that $\lim_{M \to big} \omega = -\lim_{M \to big} \frac{2mvr}{MR^2 + 2mr^2} \approx 0$ as expected.

(iii) What is the angular velocity if M = 450. kg (a weight of 1000 lb), R = 1.5 m (5 ft), m = 50. kg (a weight of 110 lb), v = 0.50 m/s (1 mile per hour) and r = 1.2 m (4 ft)?

$$\omega = -\frac{2mvr}{MR^2 + 2mr^2} = -\frac{2(50)(0.5)(1.2)}{(450)(1.5)^2 + 2(50)(1.2)^2} = -\frac{60}{1012.5 + 144} = -\frac{60}{1156.5} = -0.05188$$

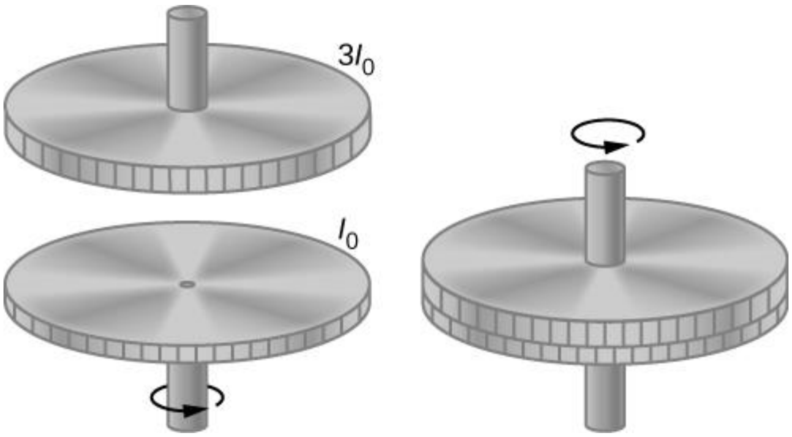
$$\boxed{\omega = -0.052\,\frac{\text{rad}}{\text{s}}}$$

(iv) Time for one rotation.

$$\omega = 2\pi f = 2\pi \frac{1}{T} \quad \Rightarrow \quad T = \frac{2\pi}{\omega} \quad \Rightarrow \quad T = \frac{2\pi}{0.05188} = 121.1 \text{ s}$$

$$\boxed{T = 2 \text{ min}}$$

N4. Coupled Flywheels. This problem is taken from the excellent online text by Professor Charles Niederriter, Gustavus Adolphus College, St. Peter, Minnesota, USA. The text is available for free at openstax, Courtesy Rice University, Houston, Texas, USA.



"A flywheel rotates without friction at an angular velocity ω_0 = 600 rev/min on a frictionless, vertical shaft of negligible rotational inertia. A second flywheel, which is at rest and has a moment of inertia three times that of the rotating flywheel, is dropped onto it (Figure). Because friction exists between the surfaces, the flywheels very quickly reach the same rotational velocity, after which they spin together. (a) Use the law of conservation of angular momentum to determine the angular velocity ω of the combination. (b) What fraction of the initial kinetic energy is lost in the coupling of the flywheels?" Mechanical Universe by Charles Niederriter.

Solution. (a) Since no external torques act on the system about the axis, we can use conservation of angular momentum.

$$L_{before} = L_{after} \quad \Rightarrow \quad I_0\omega_0 = (I_0 + 3I_0)\omega$$

$$I_0\omega_0 = 4I_0\omega \quad \Rightarrow \quad \omega_0 = 4\omega \quad \Rightarrow \quad \boxed{\omega = \frac{\omega_0}{4}}$$

The initial angular velocity is given as $\omega_0 = 600\ \dfrac{\text{rev}}{\text{min}}$, where you are expected to get it into radians per second. Technically, we are given a frequency $f_0 = 600\ \dfrac{\text{rev}}{\text{min}}$. So, let's convert.

Conversion Method 1. Our good old units conversion tricks.

$$600\ \frac{\text{rev}}{\text{min}} = 600\ \frac{\text{rev}}{\text{min}} \cdot \frac{1\text{ min}}{60\text{ s}} \cdot \frac{2\pi\text{ rad}}{1\text{ rev}} = \frac{600}{60} \cdot 2\pi \frac{\text{rad}}{\text{s}} = 10 \cdot 2\pi \frac{\text{rad}}{\text{s}} = 20\pi \frac{\text{rad}}{\text{s}}$$

Conversion Method 2. We recognize we are given the rate in the form $f_0 = 600\ \dfrac{\text{rev}}{\text{min}}$. Then

$$\omega_0 = 2\pi f_0 \quad \Rightarrow \quad \omega_0 = 2\pi \cdot 600 \frac{\text{rad}}{\text{min}} = 2\pi \cdot 600 \frac{\text{rad}}{60\text{ s}} = 2\pi \cdot 10 \frac{\text{rad}}{\text{s}} = 20\pi \frac{\text{rad}}{\text{s}}.$$

We proceed now to get the answer for the final angular velocity.

$$\omega_0 = 20\pi \ \frac{\text{rad}}{\text{s}} \quad => \quad \omega = \frac{\omega_0}{4} = \frac{20\pi}{4} \ \frac{\text{rad}}{\text{s}} = 5\pi \frac{\text{rad}}{\text{s}} \quad => \quad \boxed{\omega = 15.7 \frac{\text{rad}}{\text{s}}}$$

(b) What fraction of the initial kinetic energy is lost in the coupling of the flywheels?"

$$K_{before} = \frac{1}{2} I_0 \omega_0^2$$

$$K_{after} = \frac{1}{2}(I_0 + 3I_0)\omega^2 = \frac{1}{2}(4I_0)\omega^2 = 2I_0\omega^2$$

The change in kinetic energy is $\Delta K = K_{after} - K_{before} = 2I_0\omega^2 - \frac{1}{2} I_0 \omega_0^2$

Substitute $\omega = \frac{\omega_0}{4}$.

$$\Delta K = 2I_0 (\frac{\omega_0}{4})^2 - \frac{1}{2} I_0 \omega_0^2 \quad => \quad \Delta K = 2I_0 \frac{1}{16} \omega_0^2 - \frac{1}{2} I_0 \omega_0^2 \quad => \quad \Delta K = (\frac{1}{8} - \frac{1}{2}) I_0 \omega_0^2$$

$$\Delta K = (\frac{1-4}{8}) I_0 \omega_0^2 \quad => \quad \Delta K = -\frac{3}{8} I_0 \omega_0^2$$

The negative sign indicates we lost kinetic energy. The fraction lost is

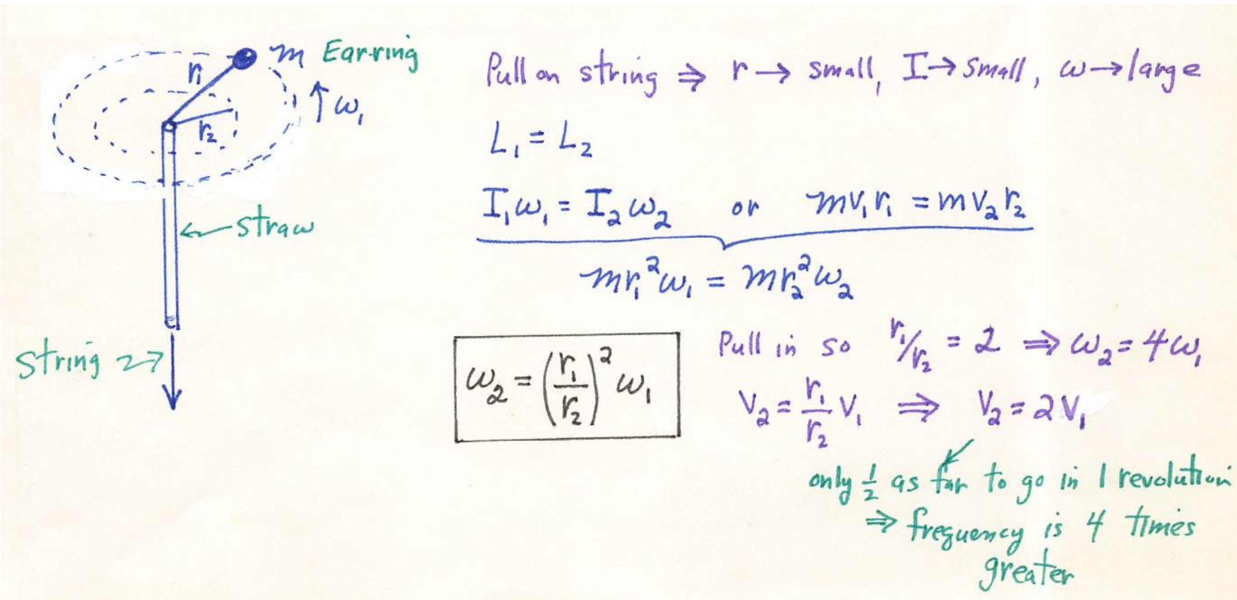
$$\frac{|\Delta K|}{K_{before}} = \frac{\frac{3}{8} I_0 \omega_0^2}{\frac{1}{2} I_0 \omega_0^2} = \frac{3}{8} \cdot \frac{2}{1} = \frac{3}{4}$$

Shortcut. Since we know $K_{before} = \frac{1}{2} I_0 \omega_0^2$ and $\omega = \frac{\omega_0}{4}$. The final kinetic energy is then

$$K_{after} = \frac{1}{2}(I_0 + 3I_0)\omega^2 = \frac{1}{2}(4I_0)\omega^2 = 2I_0\omega^2 = 2I_0 (\frac{\omega_0}{4})^2 = \frac{2}{16} I_0 \omega_0^2 = \frac{1}{8} I_0 \omega_0^2$$

$$\frac{K_{after}}{K_{before}} = \frac{\frac{1}{8} I_0 \omega_0^2}{\frac{1}{2} I_0 \omega_0^2} = \frac{1}{8} \cdot \frac{2}{1} = \frac{1}{4}$$ => We lost $\frac{3}{4}$, i.e., 75%.

N5. Centripetal Force and the Straw. A classic demonstration is shown below, where a mass m is attached to a string and twirled around in a circle with radius r_1. Then you pull on the string to decrease the radius to r_2. The moment of inertia $I = mr^2$ decreases and the mass speeds up. Angular momentum is conserved since the force you exert acts perpendicular to the motion and does not add any external torque to the motion.



Problem. See the description of the classic centripetal force demonstration above. An initial circular revolution is obtained for mass m at radius r_1 with angular velocity ω_1. The string is then pulled so that the radius is decreased to r_2 with a greater angular velocity ω_2. Let the radii be related by $r_1 = \beta r_2$, where $\beta > 1$.

(i) Angular Velocity. Find the angular velocity ω_2 at radius r_2.

(ii) Kinetic Energy. Find the kinetic energy gained as the mass goes from radius r_1 to radius r_2.

(iii) Work Energy Theorem. Show that the work done by pulling the string accounts for where the gain in kinetic energy came from.

(iv) Numerical Answers. Calculate ω_2, f_2, and ΔK when $m = 3.0\ \text{g}$, $r_1 = 20.\ \text{cm}$, $f_1 = 1.0\ \text{Hz}$, and $r_2 = 10.\ \text{cm}$. Finally give f_2.

Solution.

(i) Angular Velocity. Since there are no external torques, we begin with conservation of angular momentum.

$$L_1 = L_2 \quad => \quad I_1\omega_1 = I_2\omega_2 \quad => \quad mr_1^2\omega_1 = mr_2^2\omega_2$$

$$mr_1^2\omega_1 = mr_2^2\omega_2 \quad => \quad r_1^2\omega_1 = r_2^2\omega_2 \quad => \quad \frac{r_1^2}{r_2^2}\omega_1 = \omega_2$$

$$\boxed{\omega_2 = (\frac{r_1}{r_2})^2\omega_1}$$

With the representation $r_1 = \varepsilon r_2$, we find the following formula.

$$\omega_2 = (\frac{r_1}{r_2})^2\omega_1 = (\frac{\beta r_2}{r_2})^2\omega_1 = (\beta)^2\omega_1$$

$$\boxed{\omega_2 = \beta^2\omega_1}$$

Since $\beta > 1$, we will have $\omega_2 > \omega_1$.

(ii) Kinetic Energy. The gain in kinetic energy is given below.

$$\Delta K = K_2 - K_1 \quad => \quad \Delta K = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \quad => \quad \Delta K = \frac{1}{2}m(\omega_2 r_2)^2 - \frac{1}{2}m(\omega_1 r_1)^2$$

Substitute in $\omega_2 = \beta^2\omega_1$ and $r_2 = \frac{r_1}{\beta}$.

$$\Delta K = \frac{1}{2}m(\omega_2 r_2)^2 - \frac{1}{2}m(\omega_1 r_1)^2 \quad => \quad \Delta K = \frac{1}{2}m(\beta^2\omega_1 \cdot \frac{r_1}{\beta})^2 - \frac{1}{2}m(\omega_1 r_1)^2$$

$$\Delta K = \frac{1}{2}m(\beta\omega_1 r_1)^2 - \frac{1}{2}m(\omega_1 r_1)^2 \quad => \quad \boxed{\Delta K = \frac{1}{2}mr_1^2\omega_1^2(\beta^2 - 1)}$$

(iii) Work Energy Theorem.

$$W = \int \vec{F} \cdot \vec{dr} = \Delta K$$

The force is the centripetal force $\vec{F} = -\frac{mv^2}{r}\hat{r}$, in the opposite direction of increasing radius. Or in words, the centripetal force $\vec{F}$ is at 180° with respect to $\vec{dr}$. Therefore, the integral is

$W = \int \vec{F} \cdot \vec{dr} = -\int F dr$, where the magnitude of the force is $F = \dfrac{mv^2}{r}$.

Both the speed and radius will change as we pull in.

However, the angular momentum is a constant, which we can arrive at in two ways:

$L = I\omega = (mr^2)(\frac{v}{r}) = mrv$ or $L = rp = rmv$.

Use $L = mvr = const$ to our advantage as follows.

$$F = \frac{mv^2}{r} = \frac{m^2v^2r^2}{mr^3} = \frac{L^2}{mr^3}$$

Now it is clear where the constants are and the variable to integrate.

$$W = -\int F dr \to \int_{r_1}^{r_2} \frac{L^2}{mr^3} dr = \frac{L^2}{m} \int_{r_1}^{r_2} \frac{1}{r^3} dr$$

$$W = \frac{L^2}{m} \int_{r_1}^{r_2} \frac{1}{r^3} dr = -\frac{L^2}{m} \left[-\frac{1}{2r^2} \right]_{r_1}^{r_2}$$

$$W = \frac{L^2}{m} \frac{1}{2r^2} \Bigg|_{r_1}^{r_2} = \frac{L^2}{2m} (\frac{1}{r_2^2} - \frac{1}{r_1^2})$$

$$W = \frac{L^2}{2mr_1^2} (\frac{r_1^2}{r_2^2} - 1)$$

Now substitute $r_1 = \beta r_2$.

$$W = \frac{L^2}{2mr_1^2} (\frac{\beta^2 r_2^2}{r_2^2} - 1) \quad \Rightarrow \quad W = \frac{L^2}{2mr_1^2} (\beta^2 - 1)$$

Next substitute $L = mr^2\omega = mr_1^2\omega_1$.

$$W = \frac{(mr_1^2\omega_1)^2}{2mr_1^2} (\beta^2 - 1) \quad \Rightarrow \quad W = \frac{m^2 r_1^4 \omega_1^2}{2mr_1^2} (\beta^2 - 1) \quad \Rightarrow \quad W = \frac{mr_1^2\omega_1^2}{2} (\beta^2 - 1)$$

$$W = \frac{1}{2} m r_1^2 \omega_1^2 (\beta^2 - 1)$$

The work-energy theorem states $W = \Delta K$.

Therefore, we find that the change in kinetic energy from the work-energy theorem,

$$\Delta K = \frac{1}{2} m r_1^2 \omega_1^2 (\beta^2 - 1),$$

is exactly what we calculated for the change in kinetic energy from $\Delta K = K_2 - K_1$.

A beautiful example of physics all coming together.

Note that when you tug on the string, you tug just a little more than the centripetal force so that the radius shortens to support the new angular velocity. We saw that

$$F = \frac{mv^2}{r} = \frac{m^2 v^2 r^2}{mr^3} = \frac{L^2}{mr^3},$$

which means $F \sim \dfrac{1}{r^3}$.

The force you apply increases as $\dfrac{1}{r^3}$ as you pull the string down.

(iv) Numerical Answers. Calculate ω_2, f_2, and ΔK when $m = 3.0 \text{ g}$, $r_1 = 20. \text{ cm}$, $f_1 = 1.0 \text{ Hz}$, and $r_2 = 10. \text{ cm}$. Finally give f_2.

$$\boxed{\omega_2 = (\frac{r_1}{r_2})^2 \omega_1} \qquad \boxed{\Delta K = \frac{1}{2} m r_1^2 \omega_1^2 (\beta^2 - 1)}$$

$$\omega_2 = (\frac{r_1}{r_2})^2 \omega_1 = (\frac{r_1}{r_2})^2 2\pi f_1 = (\frac{20}{10})^2 (2\pi \cdot 1) = 4 \cdot 2\pi = 8\pi = 25.13$$

$$\boxed{\omega_2 = 25 \frac{\text{rad}}{\text{s}}}$$

The frequency f_2 is easily obtained from $\omega_2 = 8\pi \dfrac{\text{rad}}{\text{s}}$.

Each $\omega_2 = 2\pi \text{ rad}$ is a revolution. Therefore $\boxed{f_2 = 4.0 \text{ Hz}}$

For the kinetic energy $\Delta K = \frac{1}{2} m r_1^2 \omega_1^2 (\beta^2 - 1)$ we note that $\beta = \frac{r_1}{r_2} = \frac{20.0}{10.0} = 2$.

$$\Delta K = \frac{1}{2} m r_1^2 (2\pi f_1)^2 (\beta^2 - 1) = \frac{1}{2}(0.03 \text{ kg})(0.20 \text{ m})^2 (2\pi \cdot 1 \frac{1}{\text{s}})^2 (2^2 - 1)$$

$$\Delta K = (0.03 \text{ kg})(0.10 \text{ m})^2 (2\pi \frac{1}{\text{s}})^2 (4 - 1)$$

$$\Delta K = (0.03 \text{ kg})(0.01 \text{ m}^2)(4\pi^2 \frac{1}{\text{s}^2})(3)$$

$$\Delta K = (0.0003 \text{ kg} \cdot \text{m}^2)(12\pi^2 \frac{1}{\text{s}^2})$$

$$\boxed{\Delta K = 0.036 \text{ kg} \cdot \frac{\text{m}^2}{\text{s}^2}}$$

The dimensions check out: mass time velocity squared.

N6. Applying Physics to Bowling – Slipping and Rolling. When we are confronted with a physics problems in real life, it will not necessarily neatly fit into one of our chapters, but may need selected principles from various chapters. The bowling ball problem is a nice example of such a problem.

Bowling at Fort George G. Meade, Maryland, USA. My wedding party was at Ft. Meade.

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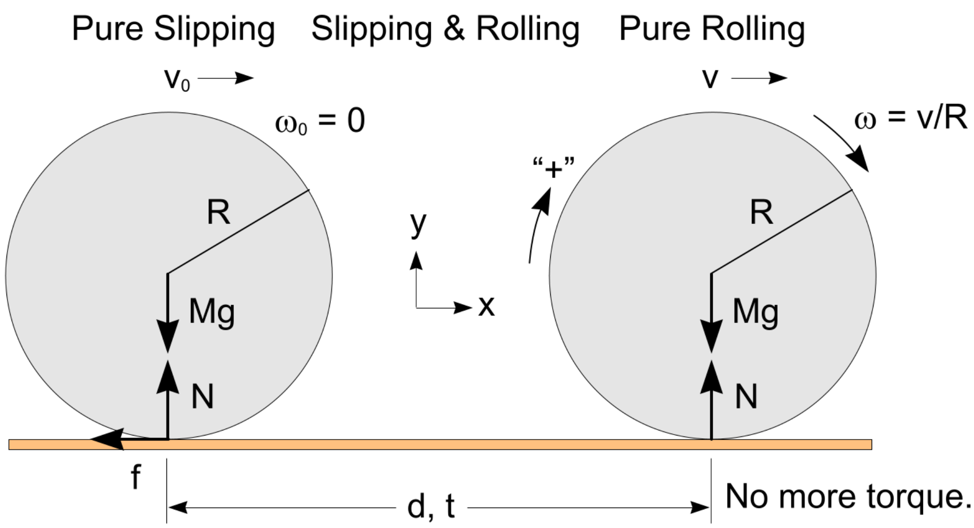
Problem. A bowling ball with mass $M = 6.0\ \text{kg}$ (weight of 13 lb) and radius $R = 10.9\ \text{cm}$ (4.3 inches) is thrown straight down the middle of a bowling alley with no spin and speed $v_0 = 7.0\ \frac{\text{m}}{\text{s}}$ (16 mph). The ball initially slips on the lane, but eventually it starts to roll. The coefficient of kinetic friction between our particular bowling ball and lane surface is $\mu = 0.20$. The length of the bowling lane from the foul line (where the ball is released) to the head pin is $18.3\ \text{m}$ (60.0 ft).

(i) The velocity v of the ball after it stops slipping. First find a general formula.
(ii) The distance d traveled during the slipping phase. First find a general formula.
(iii) The time of travel t during the slipping phase. First find a general formula.
(iv) The time T for the ball to travel from the foul line to the head pin.
(v) The total number n of spins the ball makes during its purely rolling phase.

This problem has the best of both worlds: theoretical derivation of formulas and then real-world data for numerical calculations. The first part of this problem is very common and stated very briefly as follows:

(i) A bowling ball is thrown down a bowling alley with initial speed v_0 and no spin. What is the final velocity v of the ball after it stops slipping and begins a pure roll?

That's all you get. They do not give you any more information. The problem is scary and I found it challenging the first time I encountered it.



The equations of motion throughout distance d and time t are as follows.

$$\sum F_x = -f = Ma \qquad \sum F_y = N - Mg = 0 \qquad f = \mu N$$

The three equations can be written as

$$-f = Ma$$

$$N = Mg$$

$$f = \mu N$$

Working from the bottom up as we have done often,

$$f = \mu N = \mu Mg \ .$$

Now use the first equation to find the acceleration.

$$-f = Ma \quad => \quad a = -\frac{f}{M} \quad => \quad a = -\frac{\mu Mg}{M}$$

$$\boxed{a = -\mu g}$$

We turn next to the torque equation.

$$\sum \tau = fR = I\alpha$$

For the solid bowling ball, not worrying about the three little holes for the fingers,

$$I = \frac{2}{5}MR^2 .$$

Substituting this rotational inertia into the torque equation and using $f = \mu Mg$,

$$fR = I\alpha \quad => \quad (\mu Mg)R = (\frac{2}{5}MR^2)\alpha$$

$$(\mu g)R = (\frac{2}{5}R^2)\alpha \quad => \quad (\mu g) = (\frac{2}{5}R)\alpha \quad => \quad R\alpha = \frac{5}{2}\mu g$$

$$\boxed{\alpha = \frac{5}{2}\frac{\mu g}{R}}$$

Now what?

We need to add some translational and rotational kinematics.

Translational: $a = -\mu g = \dfrac{v - v_0}{t}$

Rotational: $\alpha = \dfrac{5}{2}\dfrac{\mu g}{R} = \dfrac{\omega - \omega_0}{t} = \dfrac{\omega}{t} = \dfrac{v}{R}\dfrac{1}{t}$ since $v = \omega R$ for the rolling.

The trick now is to note that the same time t appears in each equation:

$$-\mu g = \frac{v - v_0}{t} \quad \text{and} \quad \frac{5}{2}\frac{\mu g}{R} = \frac{v}{R}\frac{1}{t}$$

Better yet, we can eliminate the entire group of factors: μgt!

The pair of equations

$$\begin{Bmatrix} -\mu g = \dfrac{v - v_0}{t} \\ \dfrac{5}{2}\dfrac{\mu g}{R} = \dfrac{v}{R}\dfrac{1}{t} \end{Bmatrix}$$

can be written as

$$\begin{Bmatrix} -\mu g t = v - v_0 \\ \dfrac{5}{2}\dfrac{\mu g t}{R} = \dfrac{v}{R} \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} \mu g t = v_0 - v \\ \dfrac{5}{2}\mu g t = v \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} \mu g t = v_0 - v \\ \mu g t = \dfrac{2}{5} v \end{Bmatrix}.$$

From these two equations we eliminate the $\mu g t$ and arrive at

$$v_0 - v = \frac{2}{5} v .$$

We want to solve for v .

$$v_0 - v = \frac{2}{5} v \quad \Rightarrow \quad v_0 = v + \frac{2}{5} v \quad \Rightarrow \quad v_0 = \frac{5}{5} v + \frac{2}{5} v \quad \Rightarrow \quad v_0 = \frac{7}{5} v$$

We finally arrive at the cool answer!

$$\boxed{v = \frac{5}{7} v_0}$$

The bowling ball has slowed down from its initial v_0 to $\frac{5}{7} v_0$.

For $v_0 = 7.0 \ \frac{\text{m}}{\text{s}}$ (16 mph), we find $v = \frac{5}{7} v_0 = \frac{5}{7} \cdot 7.0 \ \frac{\text{m}}{\text{s}}$.

$$\boxed{v = 5.0 \ \frac{\text{m}}{\text{s}}}$$

This speed is equivalent to 11 mi/h = 18 km/h = 16 ft/s.

(ii) The distance d traveled during the slipping phase. This beautiful problem not only reviews translational and rotational dynamics, but also translational and rotational kinematics. We use another kinematic equation to obtain the distance d .

The kinematic equation $2ad = v^2 - v_0^2$ in conjunction with $a = -\mu g$ will get us d.

$$2ad = v^2 - v_0^2 \quad => \quad 2(-\mu g)d = v^2 - v_0^2 \quad => \quad d = \frac{v^2 - v_0^2}{-2\mu g}$$

$$d = \frac{v_0^2 - v^2}{2\mu g}$$

Finally, we invoke $v = \frac{5}{7}v_0$. Then

$$v_0^2 - v^2 = v_0^2 - (\frac{5}{7}v_0)^2 \quad => \quad v_0^2 - v^2 = \left[1 - (\frac{5}{7})^2\right]v_0^2 \quad => \quad v_0^2 - v^2 = \left[1 - \frac{25}{49}\right]v_0^2$$

$$v_0^2 - v^2 = (\frac{49}{49} - \frac{25}{49})v_0^2 \quad => \quad v_0^2 - v^2 = \frac{24}{49}v_0^2$$

Substitution this last equation into $d = \frac{v_0^2 - v^2}{2\mu g}$ leads to

$$d = \frac{1}{2\mu g}\frac{24}{49}v_0^2 \quad => \quad d = \frac{1}{\mu g}\frac{12}{49}v_0^2$$

$$\boxed{d = \frac{12}{49}\frac{v_0^2}{\mu g}}$$

A very cool formula.

With $v_0 = 7.0 \ \frac{\text{m}}{\text{s}}$ and $\mu = 0.20$,

$$d = \frac{12}{49}\frac{v_0^2}{\mu g} = \frac{12}{49}\cdot\frac{(7.0)^2}{0.20\cdot 9.8} = \frac{12}{49}\cdot\frac{49}{0.20\cdot 9.8} = \frac{12}{0.20\cdot 9.8} = \frac{60}{9.8} = 6.122 \text{ m}$$

$$\boxed{d = 6.1 \text{ m}}$$

This distance is 100% x 6.1m/18.3m = 33% down the bowling lane.

(iii) The time of travel t during the slipping phase. We can go to another kinematic formula, one with the time. A simple one is $v = v_0 + at$.

$$v = v_0 + at \quad => \quad at = v - v_0 \quad => \quad t = \frac{v - v_0}{a}$$

Substitute in $a = -\mu g$.

$$t = \frac{v - v_0}{a} \quad => \quad t = \frac{v - v_0}{-\mu g} \quad => \quad t = \frac{v_0 - v}{\mu g}$$

Then using $v = \frac{5}{7} v_0$, the difference in velocities is

$$v_0 - v = v_0 - \frac{5}{7} v_0 = \frac{2}{7} v \text{ .}$$

With this difference in the numerator of $t = \frac{v_0 - v}{\mu g}$ we get our equation.

$$t = \frac{v_0 - v}{\mu g} \quad => \quad t = \frac{1}{\mu g} \frac{2}{7} v_0$$

$$\boxed{t = \frac{2}{7} \frac{v_0}{\mu g}}$$

All of these formulas are neat.

With $v_0 = 7.0 \ \frac{\text{m}}{\text{s}}$ and $\mu = 0.20$,

$$t = \frac{2}{7} \frac{v_0}{\mu g} = \frac{2}{7} \frac{7}{0.20 \cdot 9.8} = \frac{2}{0.20 \cdot 9.8} = \frac{10}{9.8} = 1.02 \text{ s}$$

$$\boxed{t = 1.0 \text{ s}}$$

(iv) The time T for the ball to travel from the foul line to the head pin. The distance from the foul line to the front center pin is given as 18.3 m. Therefore, the rolling distance is

$$18.3 - 6.1 = 12.2 \text{ m}$$

During the rolling phase the ball travels at $v = \frac{5}{7} v_0 = \frac{5}{7} \cdot 7.0 \ \frac{\text{m}}{\text{s}} = 5.0 \ \frac{\text{m}}{\text{s}}$. Using $d_{roll} = v t_{roll}$

$$t_{roll} = \frac{d_{roll}}{v} = \frac{12.2}{5.0} = 2.44 \text{ s}$$

The total time from the foul line to the pin is then

$$T = T_{skid} + T_{roll} = 1.0 \text{ s} + 2.4 \text{ s}$$

$$\boxed{T = 3.4 \text{ s}}$$

(v) The total number of spins the ball makes during its purely rolling phase. We have found , the rolling distance to be

$$18.3 - 6.1 = 12.2 \text{ m}.$$

The bowling ball has a radius $R = 10.9 \text{ cm}$. This radius leads to a circumference of

$$C = 2\pi R = 2\pi \cdot 10.9 \text{ cm} = 68.49 \text{ cm} = 0.685 \text{ m}.$$

The number of rotations is then

$$\frac{12.2 \text{ m}}{0.685 \text{ m}} = 17.8$$

$$\boxed{n \approx 18}$$