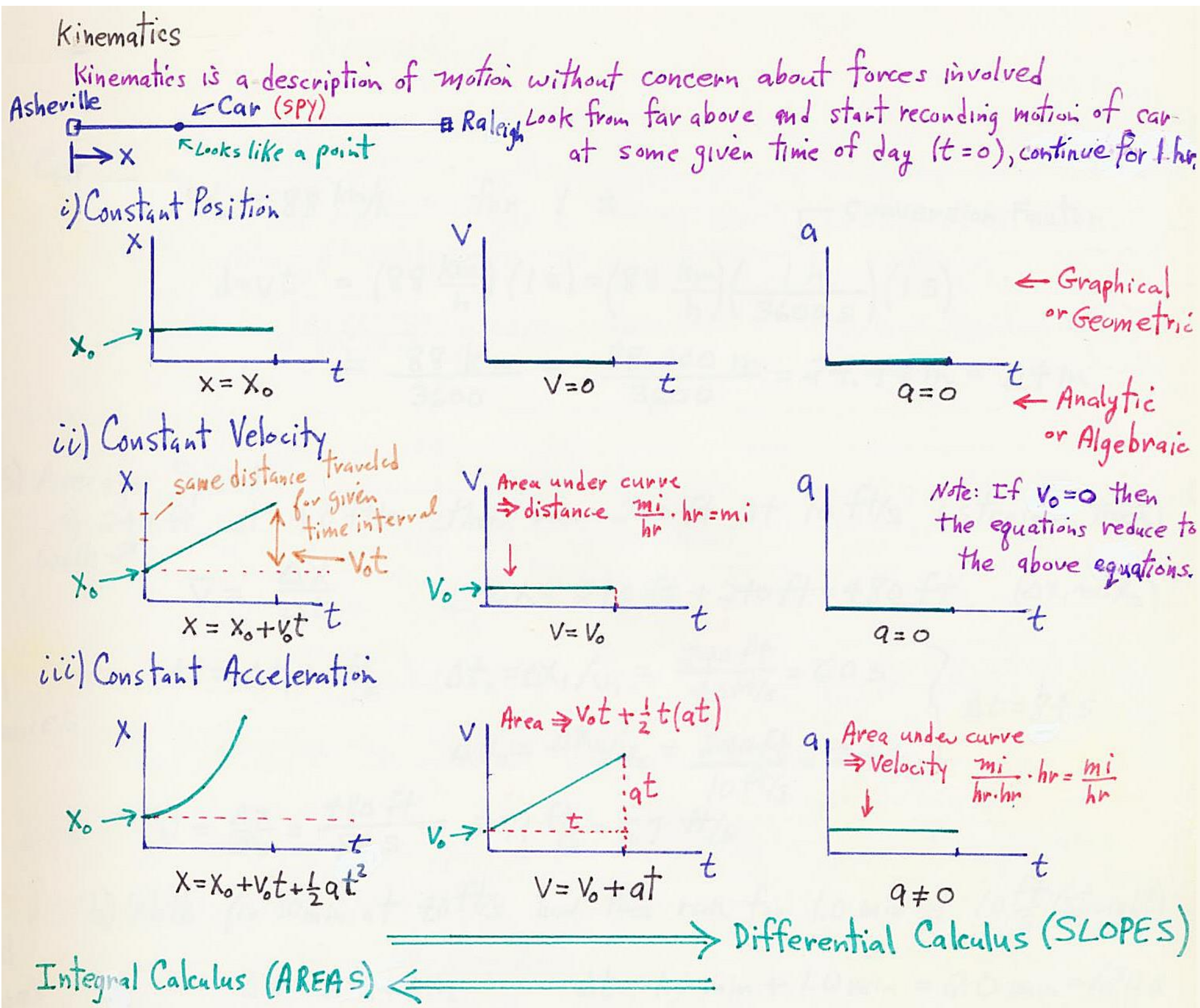
Physics I with Calculus, Prof. Ruiz (Doc), UNC-Asheville (1978-2021), DoctorPhys on YouTube
Chapter L. Moments of Inertia. **Prerequisite: Calculus I. Corequisite: Calculus II.**

L0. Calculus Review. The time has come where we are going to need to use material from your Calculus II course, which is a corequisite. I include this review because I would like you to see how a physicist looks at calculus within the context of physics. Physicists are not rigorous mathematicians and it is important for you to take calculus with mathematicians to get a solid foundation. Mathematicians often clean up after physicists. An excellent example is the work of David Hilbert (1862-1943) and the resulting Hilbert space of quantum mechanics. In a typical intro physics course, there is no time for this kind of review. But our course is not typical. So it is included. I hope you take some time to check out this section called L0. Calculus Review.



In Chapter B we saw that finding the slope is the subject of differential calculus and calculating area is the main ingredient in integral calculus. See the expression of these fundamental ideas within the context of our kinematic physics formulas in the above figure from Chapter B.

Another reason to include this short review of the foundations of differential and integral calculus is for you to have enough calculus so we can derive the moments of inertia formulas rather than just list them in a table.

1. Differential Calculus. For differential calculus we calculate a slope – the derivative – a derived function. Moving to the right in the above kinematics figure gives us slope functions.

$$v(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx(t)}{dt}$$

$$a(t) = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv(t)}{dt}$$

	Constant Function	Linear Function	Quadratic Function
Function	$x(t) = x_0$	$x(t) = x_0 + v_0 t$	$x(t) = x_0 + v_0 t + \frac{1}{2}at^2$
Slope	$v(t) = 0$	$v(t) = v_0$	$v(t) = v_0 + at$

Refer to our kinematics figure to see that the slope of a horizontal constant line is zero. For the linear function we pull off the coefficient hitting the time variable. This idea is consistent with the linear equation from algebra: $y = mx + b$. The slope is given by m. Compare $y = mx + b$ with $x(t) = x_0 + v_0 t$. The m is like the v_0. For the quadratic, the $\frac{1}{2}at^2 \to at$. The a is a constant out in front. Therefore, the slope function for t^2 is $2t$. We summarize these in the next table.

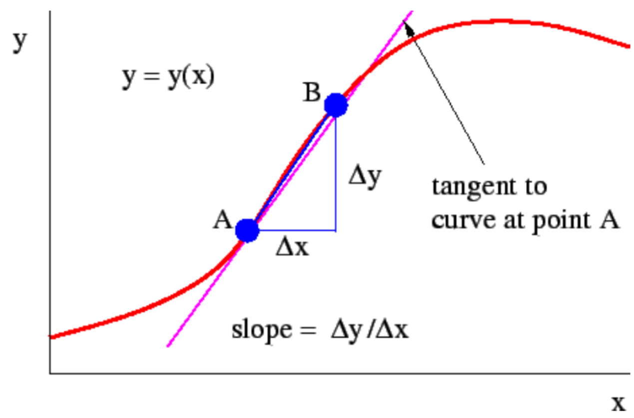
	A Simple Constant Function	Simplest Linear Function	Simplest Quadratic Function
Function	1	t	t^2
Slope	0	1	$2t$

For our math review, we would like to think in terms of $y = y(x)$ instead of $x = x(t)$. The table then becomes the one below.

	Simplest Constant Function	Simplest Linear Function	Simplest Quadratic Function
Function	1	x	x^2
Slope	0	1	$2x$

An important foundation in differential calculus is the general formula that gives the slope of a function. When you apply it to a specific function, you get the slope function, as listed in the above tables for a few specific cases.

The key idea of the slope is rise over run. It is a measure of how steep the graph is at a given point. The same principle applies to a path on a hike. How far up do you go when you take one step along the path. When you hike on flat ground, the slope is zero. See the next figure for a diagram of hiking up a steep hill.



Courtesy David J. Raymond, Wikimedia License: GNU FDL

The slope is illustrated at the left:

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x},$$

where you approach the precise result, the tangent, as you take

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \equiv \frac{dy}{dx}.$$

The new function, the derived one, is the derivative, with the following equivalent notations.

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \equiv \frac{dy}{dx} \equiv y' \equiv y'(x) \equiv f'(x) \equiv \frac{df(x)}{dx} \equiv \frac{df}{dx}$$

More details appear in the next version of the formula, one that is widespread in calculus.

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Consult the above figure to see that $\dfrac{f(x+\Delta x) - f(x)}{\Delta x} = \dfrac{\Delta y}{\Delta x} = \dfrac{\text{rise}}{\text{run}}$.

Another popular variation is one where you use h for Δx.

$$\frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We know from the physics examples above that the slope function for x^2 is $2x$. By way of a calculus review, let's use the "master slope formula" to arrive at this result. This short calculation will give us confidence in the general formula. We proceed with the formula when $f(x) = x^2$.

$$\frac{df(x)}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}=\lim_{h\to 0}\frac{(x^2+2xh+h^2)-x^2}{h}$$

$$\frac{df(x)}{dx}=\lim_{h\to 0}\frac{2xh+h^2}{h}=\lim_{h\to 0}(2x+h)=2x$$

The remarkable cancellation of the x^2 so that we can get rid of that h in the denominator results in the miracle, obtaining a finite result!

Important for physics applications is the slope function for the general case x^n, where n = 0, 1, 2, 3, … We will need the binomial theorem for the derivation.

$$(a+b)^n=a^n+na^{n-1}b+\frac{n(n-1)}{1\cdot 2}a^{n-2}b^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3+...+b^n$$

Think of apples and bananas. You will choose n pieces of fruit since there are n factors of (a + b). When multiplying these factors out, think of picking the individual "a" and "b" factors as choosing the fruit: a for apple and b for banana. There is only one way to pick all apples, i.e., one way to multiply so that all the individual factors are "a" in order to get a^n. Therefore, the coefficient in front of a^n is 1. Next we consider picking one banana and the rest apples. There are n ways to pick one banana b and the rest (n – 1) will be apples a^{n-1}. This choice is represented by the term $a^{n-1}b$ and we place the coefficient n in front for the n ways.

For the next term you pick two bananas. Once you pick the first banana out of the n possibilities, there are then n – 1 ways to pick the next banana. But since the order does not matter, we divide by 2 and arrive at $\frac{n(n-1)}{1\cdot 2}a^{n-2}b^2$ for two bananas and n – 2 apples. Can you continue along with this analogy to check the next term and start to see the pattern?

Here we go.

$$\frac{df(x)}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^n-x^n}{h}$$

From the binomial theorem $(x+h)^n=x^n+nx^{n-1}h+\frac{n(n-1)}{1\cdot 2}x^{n-2}h^2+...+h^n$.

$$\frac{df(x)}{dx}=\lim_{h\to 0}\frac{(x+h)^n-x^n}{h}=\lim_{h\to 0}\frac{\left[x^n+nx^{n-1}h+\frac{n(n-1)}{1\cdot 2}x^{n-2}h^2+...+h^n\right]-x^n}{h}$$

$$\frac{df(x)}{dx} = \lim_{h \to 0} \frac{nx^{n-1}h + \frac{n(n-1)}{1 \cdot 2} x^{n-2}h^2 + \ldots + h^n}{h}$$

$$\frac{df(x)}{dx} = \lim_{h \to 0} \left[nx^{n-1} + \frac{n(n-1)}{1 \cdot 2} x^{n-2}h + \ldots + h^{n-1} \right]$$

$$\boxed{\frac{df(x)}{dx} = nx^{n-1}}$$

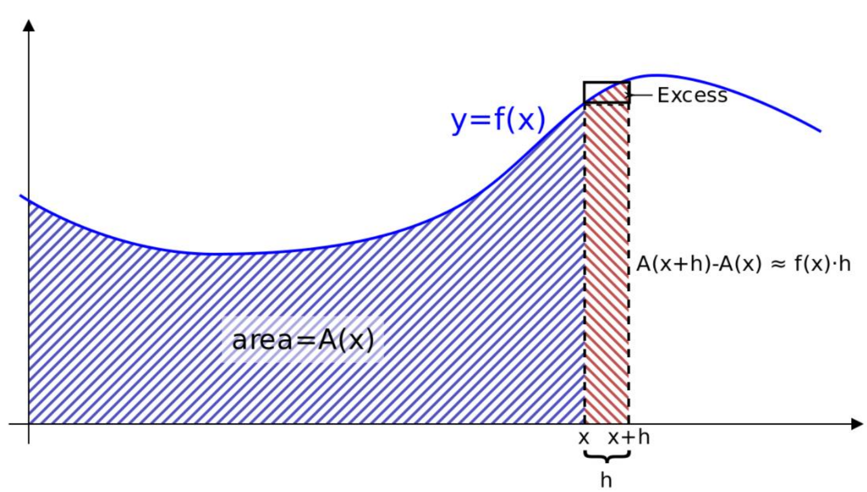
Again, we have the miraculous cancellation so that we end up with no "h" in the denominator and can therefore take the limit. The table below shows three results from this general formula, which match the few results we already know.

n	0	1	2
Function	1	x	x^2
Slope	0	1	$2x$

What about if n is a fraction or any number for that matter? That depends if the binomial theorem works in such cases. I asked my math teacher this question in college when I stopped on a campus sidewalk and saw him sitting on a bench. He said yes! Newton contributed to the generalization of the binomial theorem c. 1665. We will not dwell on the expanded power of the binomial theorem. But I will point out that when n = 1/2, we have the square root of x and we get:

$$\frac{d\sqrt{x}}{dx} = \frac{d}{dx} x^{1/2} = \frac{1}{2} x^{1/2-1} = \frac{1}{2} x^{-1} = \frac{1}{2} \frac{1}{\sqrt{x}} .$$

2. Integral Calculus. We noticed with our kinematic formulas that moving to the right in our chart gives the slope function, the derivative. Moving left gets you the area, a sort of antiderivative.



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The thin vertical red strip has approximate area

$$\Delta A \approx f(x) \cdot h ,$$

which gets better and better as the strip is taken to be thinner and thinner.

Since $\Delta A = A(x+h) - A(x)$, we can write for the small strip,

$$\Delta A = A(x+h) - A(x) \approx f(x) \cdot h \, .$$

Comparing with our kinematics chart, the area function $A(x)$ is to the left of our function $f(x)$.

Left Function	Right Function
$A(x)$	$f(x)$
Area Function	Slope Function

We now check to see if the right function $f(x)$ is the slope function for $A(x)$. Start with

$$\Delta A = A(x+h) - A(x) = f(x) \cdot h$$

and solve for $f(x)$.

$$A(x+h) - A(x) \approx f(x) \cdot h \qquad \Rightarrow \qquad f(x) \cdot h \approx A(x+h) - A(x)$$

$$f(x) \approx \frac{A(x+h) - A(x)}{h}$$

$$f(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$

The function $f(x)$ is indeed the slope function, i.e., derivative, of the area function!

This profound connection is the fundamental theorem of calculus.

If you want to get the area function $A(x)$ from $f(x)$, start with $\Delta A \approx f(x) \cdot h$ where as we noted before $h = \Delta x$.

$$\Delta A \approx f(x) \cdot \Delta x$$

The area function is then

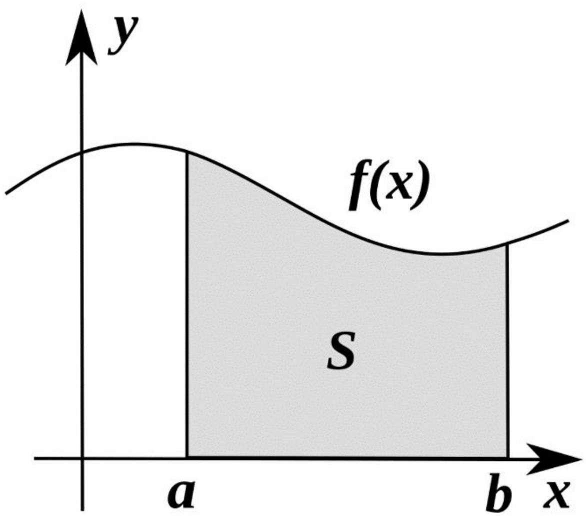
$$A(x) \approx \sum \Delta A \approx \sum f(x) \cdot \Delta x \, .$$

As we let Δx get smaller and smaller, approaching zero, we obtain an infinite number of strips. We then call Δx an infinitesimal and designate it by dx. The area $A(x)$ is still finite as we can see from its shaded portion on the above graph. The following notation means infinitesimal strips.

$$A(x) = \int f(x)dx$$

The area is called an integral and the function $f(x)$ is called the integrand. We will flip the equation to write the integral on the left side.

$$\int f(x)dx = A(x)$$



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But for a specific area S we need to ask from where to where? Say we want the area from $x = a$ to $x = b$. Then we need to subtract the total area up to $x = a$ from the total area up to $x = b$.

$$S = A(b) - A(a)$$

The following notation gets us the desired area.

$$\int_a^b f(x)dx = A(x)\Big|_a^b = A(b) - A(a)$$

If you write the non-specific form, you should include an overall constant. See the table below. The non-specific forms is called an *indefinite integral*; the specific form is called a *definite integral*.

Left Function	Right Function
$\int f(x)dx = A(x) + const$	$f(x) = \frac{d}{dx}[A(x) + const] = \frac{dA(x)}{dx}$
Area Function	Slope Function

Remember the constants x_0 and v_0 that arose when we went from right function to left function two times?

Here is the slope table again with a couple of more entries.

n	0	1	2	3	4
Function	1	x	x^2	x^3	x^4
Slope	0	1	$2x$	$3x^2$	$4x^3$

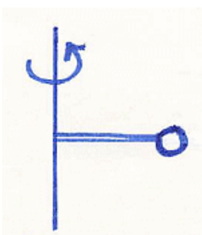
Using the rule that the area function is the reverse of the slope, we can make a similar area table being careful to add a constant in each case.

n	0	1	2	3	4
Function	1	x	x^2	x^3	x^4
Area	$x+c$	$\frac{x^2}{2}+c$	$\frac{x^3}{3}+c$	$\frac{x^4}{4}+c$	$\frac{x^5}{5}+c$

Summary.

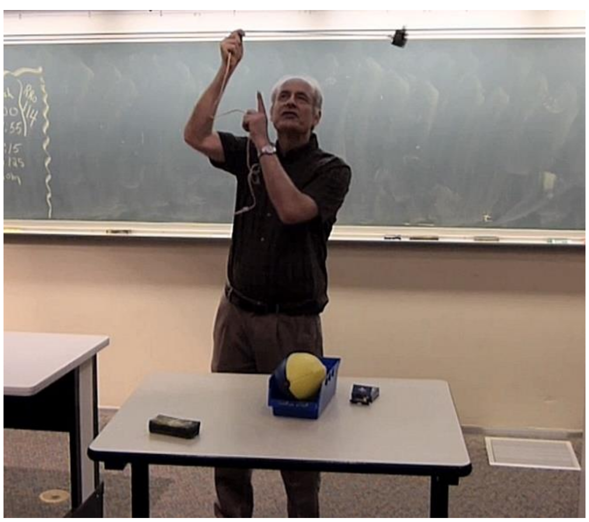
Derivative	$x^n \to nx^{n-1}$	$\frac{dx^n}{dx} = nx^{n-1}$
Integral	$x^n \to \frac{x^{n+1}}{n+1} + const$	$\int x^n dx = \frac{x^{n+1}}{n+1} + const$

L1. Moment of Inertia. Now we continue with physics. But as you will see, most of this chapter is integral calculus.



We know from the last chapter that the moment of inertia is given by the formula $I = mr^2$. We arrived at this result from considering kinetic energy.

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m(\omega r)^2 = \frac{1}{2}m\omega^2 r^2 = \frac{1}{2}mr^2\omega^2 = \frac{1}{2}I\omega^2 \quad => \quad I = mr^2$$



We have our first example. The moment of inertia for a mass a distance r from the axis of rotation is

$$I = mr^2 .$$

Let's look at it from the point of view of applying a torque. I had to apply some torque in order to get that mass twirling about my head.

$$\tau = Fr = I\alpha$$

$$\tau = mar = I\alpha$$

$$I = \frac{mar}{\alpha} \quad => \quad I = \frac{m(\alpha r)r}{\alpha} \quad => \quad \boxed{I = mr^2}$$

We arrive at the same result. For multiple point masses we sum over the individual moments of inertia.

$$\boxed{I = \sum_i I_i = \sum_i m_i r_i^2}$$

Each m_i can be written as Δm_i when part of a continuous structure we are breaking into pieces.

$$\boxed{I = \sum_i r_i^2 \Delta m_i}$$

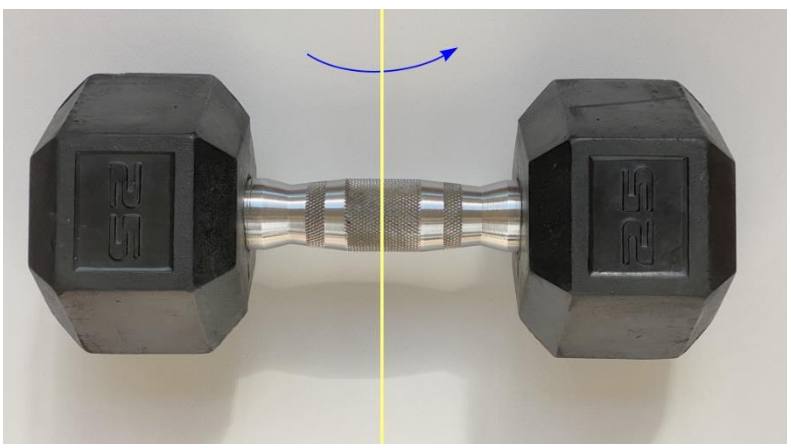
What about a continuous distribution of matter? That is when we use calculus.

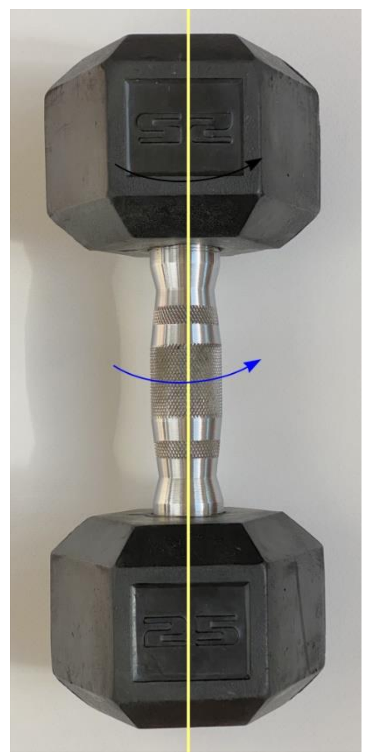
$$\boxed{I = \int r^2 dm}$$

Here is the doctorphys rule to go from a discrete sum to an integral, which is continuous.

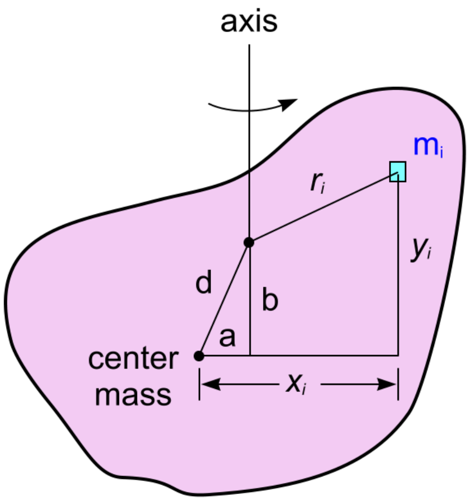
1. Rip off the “i” indexes.
2. Replace Δm with dm (or whatever the delta variable is such as $\Delta x \to dx$).
3. Change the summation sign $\sum$ into a snake $\int$, i.e., the integral sign.

Which of the two cases below represents a greater rotational inertia? For which axis is the moment of inertia greater?





L2. Two Powerful Theorems. In this section we develop two powerful theorems that will enable us to determine many moments of inertia quickly. These principles are known as the *parallel axis theorem* and the *perpendicular axis theorem*.



1. The Parallel Axis Theorem. In the figure, we have placed the axis somewhere away from the center of mass. We need to sum all the contributions of mass elements relative to this displaced axis.

$$I = \sum_i m_i r_i^2$$

We take each m_i to be a chunk with the same small mass. The distance from the axis to each little mass element is given by

$$r_i^2 = (x_i - a)^2 + (y_i - b)^2 .$$

Note that the measurements for x and y are referenced to the center of mass.

$$I = \sum_i m_i[(x_i - a)^2 + (y_i - b)^2]$$

$$I = \sum_i m_i[(x_i^2 - 2ax_i + a^2) + (y_i^2 - 2by_i + b^2)]$$

$$I = \sum_i m_i x_i^2 - 2a\sum_i m_i x_i + a^2 \sum_i m_i + \sum_i m_i y_i^2 - 2b\sum_i m_i y_i + b^2 \sum_i m_i$$

The following two terms are zero as the measurements are from the center of mass:

$$\sum_i m_i x_i = 0 \qquad \text{and} \qquad \sum_i m_i y_i = 0 .$$

We are left with

$$I = \sum_i m_i x_i^2 + a^2 \sum_i m_i + \sum_i m_i y_i^2 + b^2 \sum_i m_i ,$$

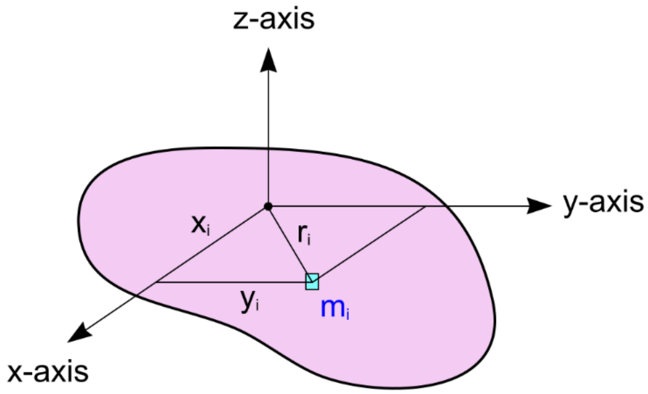
$$I = \sum_i m_i x_i^2 + Ma^2 + \sum_i m_i y_i^2 + Mb^2 ,$$

$$I = \sum_i m_i (x_i^2 + y_i^2) + M(a^2 + b^2)$$

$$\boxed{I = I_{cm} + Md^2}$$

In words, the parallel axis theorem states that the moment of inertia about an arbitrary point is equal to the sum of the moment of inertia about the center of mass plus the mass times the square of the distance to the arbitrary axis from the center of mass axis.

2. The Perpendicular Axis Theorem. This theorem is easier to derive.



We start with the moment of inertia about the z-axis,

$$I_z = \sum_i m_i r_i^2$$

and note that

$$r_i^2 = x_i^2 + y_i^2 .$$

Substituting this equation into the first gives

$$I_z = \sum_i m_i r_i^2 = \sum_i m_i (x_i^2 + y_i^2) .$$

$$I_z = \sum_i m_i x_i^2 + \sum_i m_i y_i^2$$

The first term $\sum_i m_i x_i^2 = I_y$ since distances x_i are from the y-axis.

The second term is $\sum_i m_i y_i^2 = I_x$ since distances y_i are from the x-axis.

$$\boxed{I_z = I_x + I_y}$$

In words, the perpendicular axis theorem states that the moment of inertia about one axis is equal to the sum of the moments of inertia about the other two axes. We made no mention of the center of mass for this theorem. Therefore, the z-axis can be placed anywhere.

L3. The Moment of Inertia of a Hoop. The hoop has been known for ages. In the photo below you find a specific hoop called the hula hoop. It came out when I was a boy, in the year 1958 and everyone went wild. This plastic toy version of the hoop is still very popular.

Courtesy Quinn Dombrowski
flickr, Attribution-ShareAlike

1. Hoop About Center of Mass. Here we need to imagine some very thin massless spokes that can connect the rim to the axis. Let our hoop have mass M and radius R. Since all the mass elements that make up the hoop are the same distance R from the center, we have

$$I_{cm} = \sum_i m_i r_i^2 = \sum_i m_i R^2$$

$$I_{cm} = R^2 \sum_i m_i$$

$$I_{cm} = R^2 M \quad \Rightarrow \quad \boxed{I_{cm} = MR^2}$$

This result is the same as that for a small single mass a distance R from the axis. The twirling mass example we first did and the hula loop give the same result if the masses are the same.

2. Hoop About the Edge. What is the moment of inertia of the hoop about its edge?

Courtesy North Charleston
flickr, Attribution-ShareAlike

We can use the parallel axis to shift the axis from the center of mass to the edge.

$$I = I_{cm} + Md^2$$

$$I_{edge} = MR^2 + MR^2$$

$$\boxed{I_{edge} = 2MR^2}$$

Courtesy Nathan Rupert flickr, License: Attribution-NonCommercial-NoDerivs

The second photo is another example of the hoop about an axis at the edge of the hoop.

3. The Spinning Ring. The next example is the hoop or a ring standing and undergoing a spin.

Courtesy Bence Fördős, flickr
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The perpendicular axis theorem comes in handy for this problem.

$$I_z = I_x + I_y$$

The moment of inertia about the z-axis, defined as the axis from which all the material is at a distance R is simply

$$I_z = MR^2 .$$

The moments of inertia about the two perpendicular axes are equal to each other due to symmetry. Call each of these I. Then,

$$I_z = I_x + I_y = 2I .$$

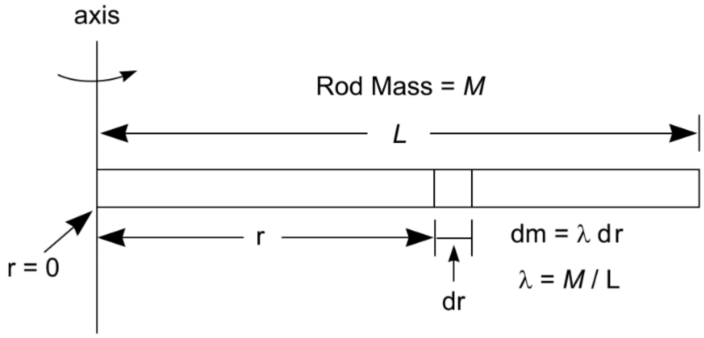
But $I_z = MR^2$. Therefore

$$MR^2 = 2I$$

The moment of inertia for the spinning ring in the figure is then give as below.

$$\boxed{I = \frac{1}{2}MR^2}$$

L4. Moment of Inertia for a Rod. First, we consider the moment of inertia for a rod about its end as illustrated in the figure below. We start with the moment of inertia formula.



$$I = \int r^2 dm$$

It is assumed that the rod has uniform density. We would like to introduce the linear density $\lambda = \frac{M}{L}$. Then, the small

$$dm = \lambda dr .$$

The integral we need to do is $I = \int r^2 dm = \int_0^L r^2 \lambda dr = \lambda \int_0^L r^2 dr$.

$$I = \lambda \int_0^L r^2 dr = \lambda \left. \frac{r^3}{3} \right|_0^L = \lambda \left[\frac{L^3}{3} - \frac{0^3}{3} \right] = \lambda \frac{L^3}{3} = \frac{M}{L}\frac{L^3}{3} \quad \Rightarrow \quad \boxed{I = \frac{1}{3}ML^2}$$

What about the center, i.e., the center of mass? The moment of inertia of the center of mass appears in our parallel axis equation below as the unknown. We are $d = L/2$ from the center.

$$I_{end} = I_{cm} + Md^2$$

$$\frac{1}{3}ML^2 = I_{cm} + M(\frac{L}{2})^2$$

$$I_{cm} = \frac{1}{3}ML^2 - \frac{1}{4}ML^2$$

$$I_{cm} = (\frac{1}{3} - \frac{1}{4})ML^2$$

$$I_{cm} = (\frac{4-3}{12})ML^2$$

$$I_{cm} = (\frac{1}{12})ML^2$$

$$\boxed{I_{cm} = \frac{1}{12}ML^2}$$

A baton is twirled at its center and is approximately a rod.

Baton Twirlers Courtesy Brian Leon, flickr
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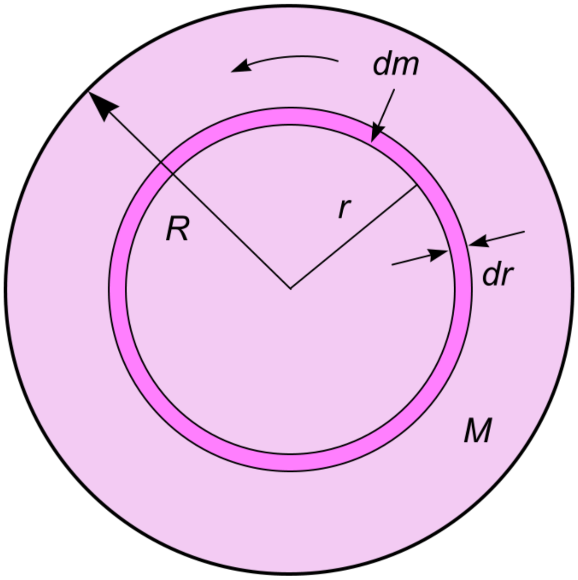
L5. The Moment of Inertia for a Disk.



1. About the Center of Mass. For the continuous uniform disk, we use

$$I = \int r^2 dm .$$

It is convenient to define the areal density

$$\sigma = \frac{M}{\pi R^2} .$$

Then the mass at a distance r from the axis is the mass in the ribbon with circumference $2\pi r$ and thickness dr. The mass in this ribbon is

$$dm = \sigma dA = \sigma(2\pi r dr)$$, giving

$$I = \int r^2 2\pi\sigma r dr ,$$

We need to integrate from $r = 0$ to $r = R$ in order to sum up all the ribbon contributions.

$$I = \int_0^R r^2 2\pi\sigma r dr = 2\pi\sigma\int_0^R r^3 dr = 2\pi\sigma \left.\frac{r^4}{4}\right|_0^R = 2\pi\sigma(\frac{R^4}{4} - \frac{0^4}{4})$$

$$I = 2\pi\sigma\frac{R^4}{4} = \pi\sigma\frac{R^4}{2}$$

$$I = \pi\frac{M}{\pi R^2}\frac{R^4}{2}$$

$$I = \pi\frac{M}{\pi}\frac{R^2}{2}$$

$$\boxed{I = \frac{1}{2}MR^2}$$

2. About the Edge.

Courtesy vinylmeister, flickr
Led Zeppelin II Vinyl Record
Attribution-NonCommercial

Drill a small hole near the top edge of the vinyl record at the left. Then place this top hole over a nail in a wall so that the nail is the axis about which the disk will rotate back and forth.

The moment of inertia of the disk about this edge is easily obtained from the parallel axis theorem $I = I_{cm} + Md^2$.

$$I_{edge} = \frac{1}{2}MR^2 + Md^2 = \frac{1}{2}MR^2 + MR^2 \quad => \quad \boxed{I_{edge} = \frac{3}{2}MR^2}$$

3. The Spinning Coin.

Courtesy Sergio Boscaino
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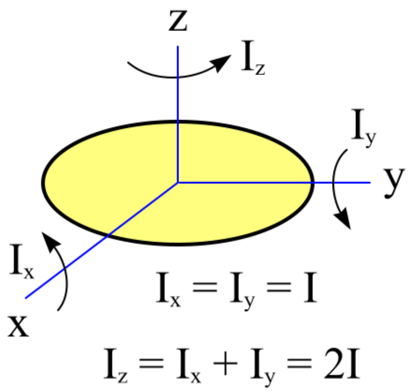
The moment of inertia for the horizontal disk about a vertical z-axis through the center of mass was found above in our first calculation for this section,

$$I_{cm} = I_z = \frac{1}{2}MR^2$$

From the perpendicular axis theorem,

$$I_z = I_x + I_y .$$



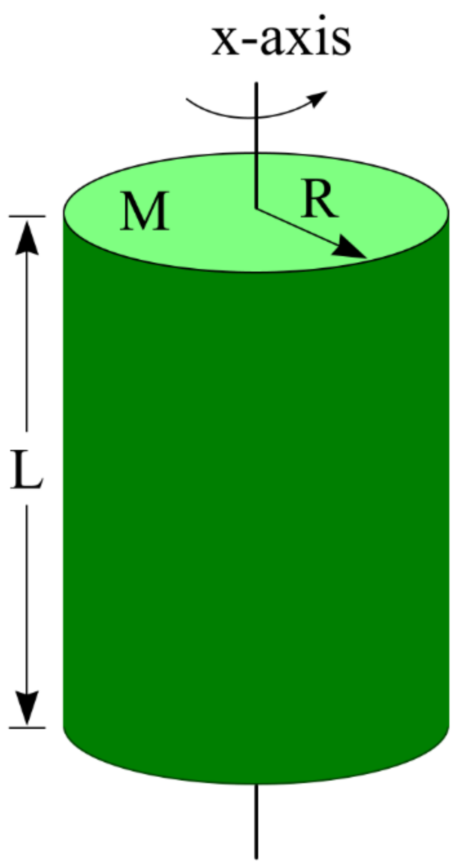
We want either I_x or I_y. But these two moments of inertia are equal due to the symmetry. Refer to the figure at the left. Therefore, our answer for the spinning coin is

$$I_x = I_y = \frac{1}{2} I_z = \frac{1}{2}(\frac{1}{2} MR^2)$$

$$\boxed{I_x = I_y = \frac{1}{2} I_z = \frac{1}{4} MR^2}$$

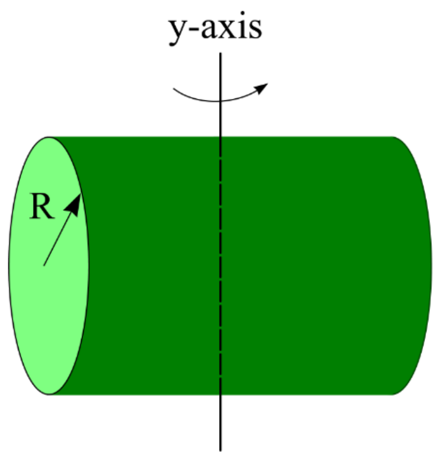
L6. Moment of Inertia for a Cylinder.

1. About the Axis Through the Long Center. The integration for the moment of inertia about the x-axis is essentially the same as that for the disk. The answer is



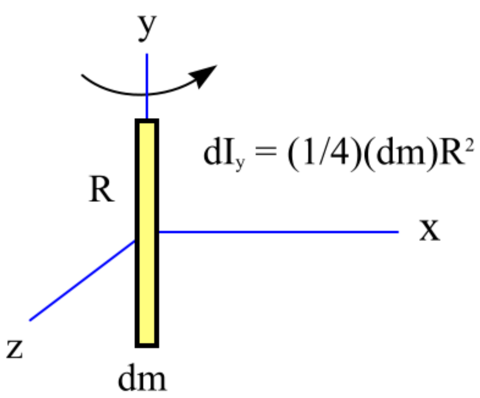
$$\boxed{I_x = \frac{1}{2} MR^2}$$

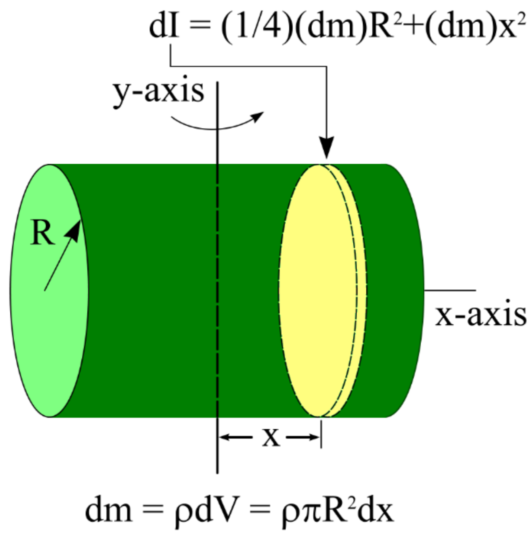
2. About the Axis Through the Short Center. The tough case is the axis below, i.e., about the y-axis.



We definitely do not want to try to integrate this one from scratch. Instead, we will first break up the cylinder into thin disks. Then we will use the parallel axis theorem to obtain its contribution to the moment of inertia. See the yellow slice in each of the figures below. For a vertical slice at the origin (see the left figure below),

$$dI_y = \frac{1}{4}(dm)R^2.$$





We shift this slice to the right using the parallel axis theorem.

$$dI = \frac{1}{4}(dm)R^2 + (dm)x^2$$

Then we use $dm = \rho\pi R^2 dx$ where the density $\rho = \frac{M}{\pi R^2 L}$. Remember that the length of our cylinder is L and the total volume of our cylinder is $\pi R^2 L$. We are almost ready for the integration.

$$dI = \frac{1}{4}(dm)R^2 + (dm)x^2 \quad \Rightarrow \quad dI = (\frac{R^2}{4} + x^2)dm \quad \Rightarrow \quad dI = (\frac{R^2}{4} + x^2)\pi\rho R^2 dx$$

We integrate from -L/2 to L/2 or double the integral from 0 to L/2. The integral is

$$I = 2\int_0^{L/2} (\frac{R^2}{4} + x^2)\rho\pi R^2 dx \text{ with } \rho = \frac{M}{\pi R^2 L}.$$

$$I = 2\rho\pi R^2 \int_0^{L/2} (\frac{R^2}{4} + x^2)dx \quad \Rightarrow \quad I = 2\rho\pi R^2 (\frac{R^2}{4}x + \frac{x^3}{3})\Big|_0^{L/2} \quad \Rightarrow \quad I = 2\rho\pi R^2 (\frac{R^2}{4}\frac{L}{2} + \frac{1}{3}\frac{L^3}{8})$$

$$I = 2\frac{M}{\pi R^2 L}\pi R^2 (\frac{R^2}{4}\frac{L}{2} + \frac{1}{3}\frac{L^3}{8}) \quad \Rightarrow \quad I = \frac{M}{L}(\frac{R^2}{4}L + \frac{1}{3}\frac{L^3}{4})$$

$$\boxed{I = \frac{1}{4}MR^2 + \frac{1}{12}ML^2}$$

3. About the Edge. We can arrive at this result in two ways.

Method 1. Integration. We redo the previous integral without the 2 in front and go from x = 0 to x = L.

$$I = \int_0^L (\frac{R^2}{4} + x^2)\rho\pi R^2 dx \text{ with } \rho = \frac{M}{\pi R^2 L}.$$

$$I = \rho\pi R^2 \int_0^L (\frac{R^2}{4} + x^2)dx \quad => \quad I = \rho\pi R^2 (\frac{R^2}{4}x + \frac{x^3}{3})\Big|_0^L \quad => \quad I = \rho\pi R^2 (\frac{R^2}{4}L + \frac{1}{3}L^3)$$

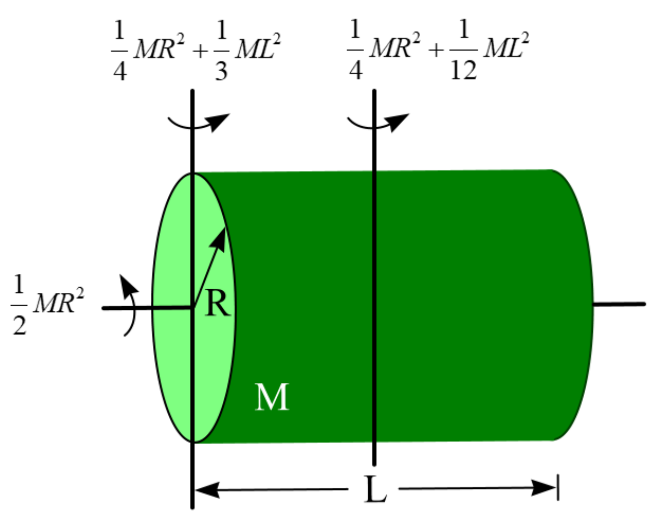
$$I = \frac{M}{\pi R^2 L}\pi R^2 (\frac{R^2}{4}L + \frac{1}{3}L^3) \quad => \quad I = \frac{M}{L}(\frac{R^2}{4}L + \frac{1}{3}L^3) \quad => \quad \boxed{I = \frac{1}{4}MR^2 + \frac{1}{3}ML^2}$$

Method 2. Parallel Axis Theorem.

$$I = I_{cm} + Md^2 = (\frac{1}{4}MR^2 + \frac{1}{12}ML^2) + M(\frac{L}{2})^2 = \frac{1}{4}MR^2 + ML^2(\frac{1}{12} + \frac{1}{4})$$

$$I = \frac{1}{4}MR^2 + ML^2(\frac{1+3}{12}) = \frac{1}{4}MR^2 + ML^2(\frac{4}{12}) = \frac{1}{4}MR^2 + \frac{1}{3}ML^2$$

A summary of all three cases is below.



Note the combination of disk and rod behavior!

Take the radius to be small and negligible and you have the results for the rod!
If the length is small an negligible, you have the results for the disk.

L7. Hypnosis. This problem involves finding the moment of inertia for a compound system.

Courtesy geralt
pixabay.com
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A chain is connected to a watch and the watch swings back and forth in the first figure.

In the lower figure, a rod is connected to a disk.

What is the moment of inertia about the pivot point?

Courtesy Joshua Kenney, flickr
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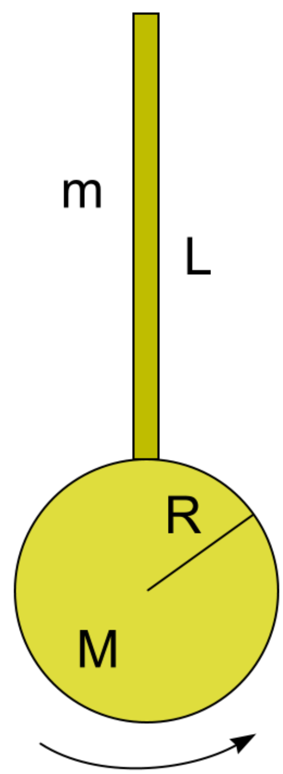
We will model these examples with a rod and a disk connected as shown below.



The rod connects to top of the disk. The moment of inertia about the top pivot point is equal to the sum of the moments of inertia for the rod and disk.

$$I = I_{rod} + I_{disk}$$

The rod is swinging by its end, while the center of mass for the disk is displaced from the pivot by $L+R$. From the last section, we know the moment of inertia for a rod swinging by its end. But I will assume we do not know it and work from the center of mass using the parallel axis theorem. Center of mass moments of inertia are easily found in tables.

$$I_{rod} = I_{cm} + md^2 = \frac{1}{12}mL^2 + m(\frac{L}{2})^2 = (\frac{1}{12} + \frac{1}{4})mL^2 = (\frac{1+3}{12})mL^2 = \frac{4}{12}mL^2 = \frac{1}{3}mL^2$$

$$I_{disk} = I_{cm} + Md^2 = \frac{1}{2}MR^2 + M(L+R)^2$$

$$I = I_{rod} + I_{disk} = \frac{1}{3}mL^2 + \frac{1}{2}MR^2 + M(L+R)^2$$

Find I in terms of M and R when $m = \dfrac{M}{4}$ and $L = 2R$.

Start with $I = \dfrac{1}{3}mL^2 + \dfrac{1}{2}MR^2 + M(L+R)^2$.

$$I = \frac{1}{3}\frac{M}{4}(2R)^2 + \frac{1}{2}MR^2 + M(2R+R)^2$$

$$I = \frac{1}{3}MR^2 + \frac{1}{2}MR^2 + M(4R^2 + 4R^2 + R^2)$$

$$I = (\frac{1}{3} + \frac{1}{2} + 4 + 4 + 1)MR^2$$

$$I = (\frac{2+3}{6} + 9)MR^2$$

$$I = (\frac{5}{6} + 9)MR^2$$

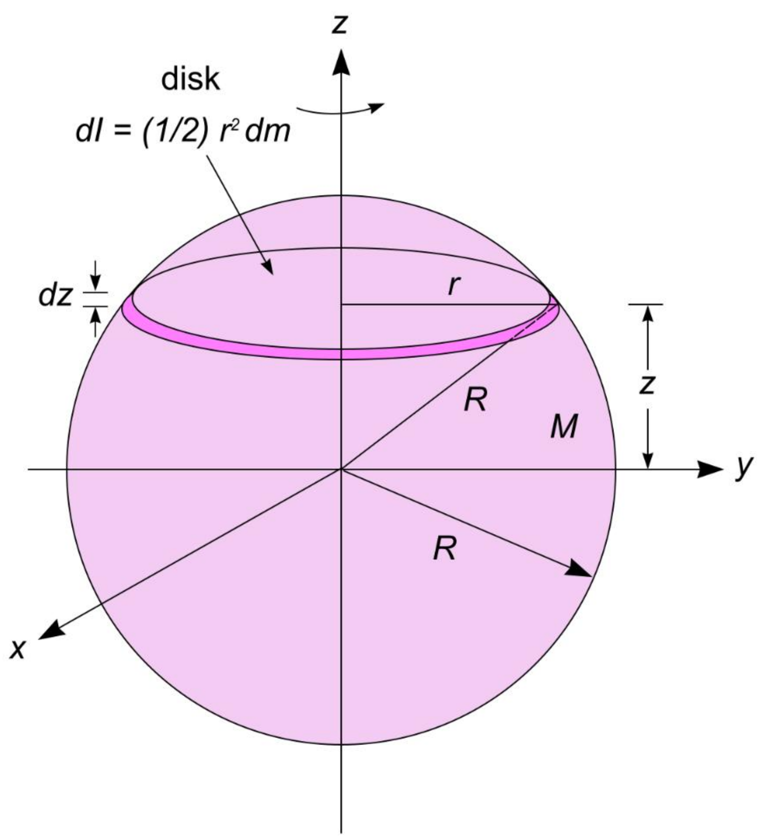
$$I = (\frac{5+54}{6})MR^2$$

$$\boxed{I = \frac{59}{6}MR^2}$$

Later in our course we will learn how to use dynamics in order to calculate the time it will take for objects to swing back and forth like a pendulum.

L8. The Moment of Inertia for a Solid Sphere. Next is the uniform solid sphere. See the figure for the parameters.

1. About the Center of Mass.



We build up the sphere from the disks for which we know the answer.

$$I_{disk} = \frac{1}{2} M R_{disk}^2$$

See the thin disk of radius r in the figure. The moment of inertia about the z-axis for this thin disk is

$$dI = \frac{1}{2} r^2 dm\,.$$

It is convenient to introduce a volume mass density using the total mass of the sphere M and the volume of the sphere $V = \frac{4}{3}\pi R^3$.

$$\rho = \frac{M}{V} = \frac{M}{4\pi R^3/3} = \frac{3M}{4\pi R^3}\,.$$

The contribution that the thin disk makes to the moment of inertia is

$$dI = \frac{1}{2} r^2 dm = \frac{1}{2} r^2 \rho dV\,,$$

where $dV = \pi r^2 dz$ is the volume of the thin plate.

$$dI = \frac{1}{2} r^2 dm = \frac{1}{2} r^2 (\rho dV) = \frac{1}{2} r^2 \rho(\pi r^2 dz) = \frac{1}{2} \rho \pi r^4 dz$$

Before we integrate, we need to express r in terms of z using $R^2 = z^2 + r^2$.

Then $r^2 = R^2 - z^2$ and $r^4 = (R^2 - z^2)^2 = R^4 - 2R^2 z^2 + z^4$, giving

$$dI = \frac{1}{2}\rho\pi r^4 dz = \frac{1}{2}\rho\pi(R^4 - 2R^2z^2 + z^4)dz\,.$$

Now we are ready to integrate. We can integrate from $z = -R$ to $z = +R$ or, since the northern and southern hemispheres contribute equally, we can double the result from $z = 0$ to $z = +R$.

$$I = 2\int_0^R \frac{1}{2}\rho\pi(R^4 - 2R^2z^2 + z^4)dz$$

$$I = \rho\pi\int_0^R (R^4 - 2R^2z^2 + z^4)dz$$

$$I = \rho\pi\left[\int_0^R R^4 dz - 2\int_0^R R^2z^2 dz + \int_0^R z^4 dz\right]$$

$$I = \rho\pi\left[R^4\int_0^R dz - 2R^2\int_0^R z^2 dz + \int_0^R z^4 dz\right]$$

$$I = \rho\pi\left[R^4 z\Big|_0^R - 2R^2\,\frac{z^3}{3}\Big|_0^R + \frac{z^5}{5}\Big|_0^R\right]$$

$$I = \rho\pi\left[R^5 - \frac{2}{3}R^5 + \frac{1}{5}R^5\right]$$

$$I = \rho\pi R^5(1 - \frac{2}{3} + \frac{1}{5})$$

$$I = \rho\pi R^5(\frac{15 - 10 + 3}{15})$$

$$I = \rho\pi R^5(\frac{8}{15})$$

Now enter the density $\rho = \dfrac{M}{V} = \dfrac{M}{\dfrac{4\pi R^3}{3}}$.

$$I = \frac{M}{\frac{4}{3}\pi R^3}\pi R^5(\frac{8}{15})$$

$$I = \frac{3}{4}MR^2(\frac{8}{15})$$

$$\boxed{I = \frac{2}{5}MR^2}$$

Note how the dimensions check out.

Bowling. The solid bowling ball rotates about its central axis as it rolls.
Courtesy Jim Pennucci, flickr, License: [Attribution 2.0 Generic]

Let's find the kinetic energy of a rolling bowling ball using our $I = \frac{2}{5}MR^2$.

$$K = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}Mv^2 + \frac{1}{2}\frac{2}{5}MR^2\omega^2 = \frac{1}{2}Mv^2 + \frac{1}{2}\frac{2}{5}MR^2(\frac{v}{R})^2 = \frac{1}{2}\frac{7}{5}Mv^2 = \frac{7}{10}Mv^2$$

$$\boxed{K = \frac{7}{10}Mv^2}$$

2. Solid Sphere Displaced from the Center of Mass. The solid sphere of mass M and radius R is now displaced so that the center of the sphere is at a distance from the pivot equal to the radius R plus the length L of the wire attached to it. For this problem, you can neglect the weight of the wire compared to the massive solid sphere. The length L is measured for this problem from the top of the solid sphere to the pivot point at the ceiling. (i) What is the moment of inertia? (ii) What is the moment of inertia for the special case $L = 0$? The length of the pendulum as we would define it in physics is given by the length $L + R$.

Courtesy Phil Roman, flickr, License: Attribution-noncommercial-NoDerivs
The Franklin Institute, a Science Museum in Philadelphia, Pennsylvania, USA

(i) Use the parallel axis theorem.

$$I = I_{cm} + Md^2 \quad \Rightarrow \quad \boxed{I = \frac{2}{5}MR^2 + M(L+R)^2}$$

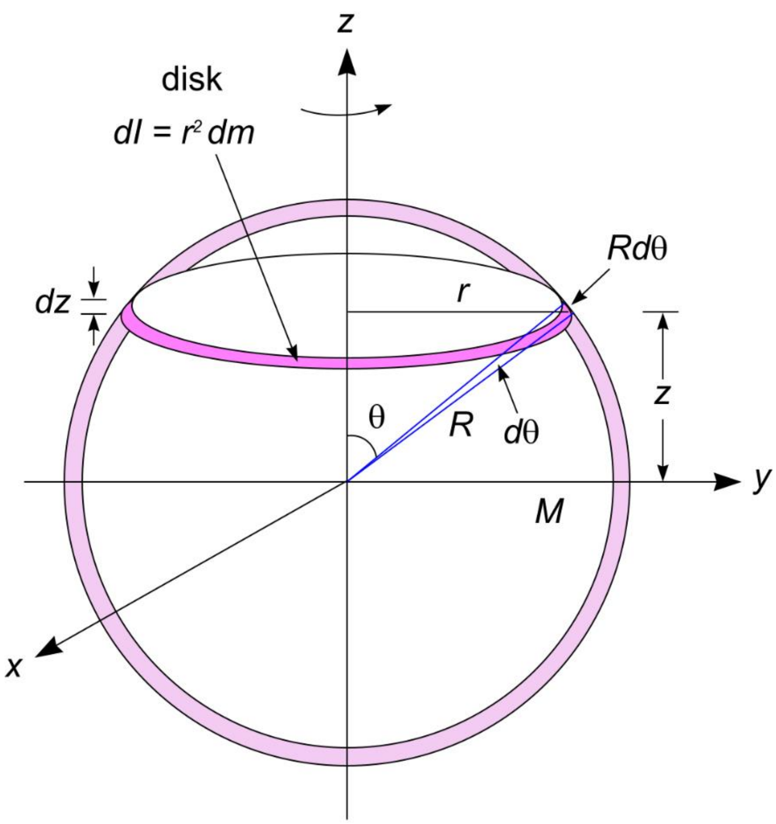
(ii) For $L = 0$,

$$I = \frac{2}{5}MR^2 + M(L+R)^2 \rightarrow \frac{2}{5}MR^2 + M(0+R)^2 = (\frac{2}{5}+1)MR^2 = \frac{7}{5}MR^2$$

$$\boxed{I = \frac{7}{5}MR^2}$$

L9. The Moment of Inertia for a Spherical Shell.

1. About the Center of Mass. We build up the shell from the ribbons where each mass element is a distance r from the z-axis.



$$dI_{ribbon} = r^2 dm_{ribbon}$$

See the thin ribbon of radius r in the figure. The mass for this thin disk is

$$dm = \sigma dA,$$

where σ is the surface density and dA is the area element. The density can be taken as a surface density since the shell is very thin. This surface density is $\sigma = \dfrac{M}{4\pi R^2}$, where $4\pi R^2$ is the total surface area A_{sphere} of the spherical shell. Here is how you can quickly arrive at this total surface area for a sphere.

$$V_{sphere} = \frac{4}{3}\pi R^3 \;\;=>\;\; dV_{sphere} = \frac{4}{3}\pi(3R^2 dR) \;\;=>\;\; dV_{sphere} = 4\pi R^2 dR = A_{sphere} dR \;\;=>\;\; A_{sphere} = 4\pi R^2$$

Summary of our results so far:

$$dI = r^2 dm, \quad dm = \sigma dA, \quad \sigma = \frac{M}{4\pi R^2}.$$

By the way, my preference is to use the symbol λ for linear density (lines), σ for surface density (areas), and ρ for solids (volumes). We need the area element of the ribbon: $dA = (2\pi r)(Rd\theta)$. Putting it all together,

$$dI = r^2 dm = r^2 \sigma dA = r^2 \sigma (2\pi r)(Rd\theta) = 2\pi\sigma r^3 R d\theta,$$

leaving the density as σ for now to reduce on writing things out. We need to decide on an integration variable and get everything in terms of that variable. I would like to use the angle variable θ. So we substitute $r = R\sin\theta$.

$$dI = 2\pi\sigma r^3 R d\theta = 2\pi\sigma (R\sin\theta)^3 R d\theta$$

$$dI = 2\pi\sigma R^3 (\sin^3\theta) R d\theta$$

$$dI = 2\pi\sigma R^4 \sin^3\theta d\theta$$

We can substitute in for the density now,

$$dI = 2\pi \frac{M}{4\pi R^2} R^4 \sin^3\theta d\theta ,$$

$$dI = \frac{MR^2}{2} \sin^3\theta d\theta .$$

We are ready for the integral, where we can integrate from $\theta = 0$ to $\theta = \pi$. But since the northern hemisphere will contribute equally as the southern hemisphere due to the symmetry, we can do twice the integration from $\theta = 0$ to $\theta = \frac{\pi}{2}$.

$$I = 2\frac{MR^2}{2}\int_0^{\pi/2} \sin^3\theta d\theta$$

$$I = MR^2 \int_0^{\pi/2} \sin^3\theta d\theta$$

Now it is time for some integration technique or tricks to find $I_s = \int_0^{\pi/2} \sin^3\theta d\theta$.

$$I_s = \int_0^{\pi/2} \sin^3\theta d\theta = \int_0^{\pi/2} \sin^2\theta \sin\theta d\theta$$

$$I_s = \int_0^{\pi/2} (1-\cos^2\theta)\sin\theta d\theta$$

Let $u = \cos\theta$. Then, $du = -\sin\theta d\theta$, where as

$0 \to \theta \to \frac{\pi}{2}$, for the u-variable, $1 \to u \to 0$.

Transforming to the new variable u,

$$I_s = \int_1^0 (1-u^2)(-du).$$

Now switch the integration limits, introducing a minus sign.

$$I_s = \int_1^0 (1-u^2)(-du) \quad \Rightarrow \quad I_s = \int_0^1 (1-u^2)du \quad \Rightarrow \quad I_s = (u-\frac{u^3}{3})\Big|_0^1 = 1-\frac{1}{3} = \frac{2}{3}$$

Then, $I = MR^2 \int_0^{\pi/2} \sin^3\theta d\theta = MR^2 I_s$ leads to

$$I = MR^2 \frac{2}{3}$$

$$\boxed{I = \frac{2}{3}MR^2}$$

The units check out.

Here is another way to get this result, along the lines of the trick we did for the volume to get the surface area. The volume trick is reproduced below so you can review it.

$$V_{sphere} = \frac{4}{3}\pi R^3 \;\Rightarrow\; dV_{sphere} = \frac{4}{3}\pi(3R^2 dR) \;\Rightarrow\; dV_{sphere} = 4\pi R^2 dR = A_{sphere} dR \;\Rightarrow\; A_{sphere} = 4\pi R^2$$

We use our moment of inertia for the solid sphere and pull off the same trick to get the shell.

$$I_{sphere} = \frac{2}{5}MR^2$$

But we first need to flush out all the R-dependence.

$$I_{sphere} = \frac{2}{5}MR^2 = \frac{2}{5}\rho(\frac{4}{3}\pi R^3)R^2 = \frac{8}{15}\rho\pi R^5$$

Now we are ready for the trick.

$$dI_{sphere} = d(\frac{8}{15}\rho\pi R^5) = \frac{8}{15}\rho\pi d(R^5) = \frac{8}{15}\rho\pi(5R^4 dR)$$

$$dI_{sphere} = \frac{8}{3}\rho\pi R^4 dR$$

$$dI_{sphere} = I_{shell} dR$$

$$I_{shell} = \frac{8}{3}\rho\pi R^4$$

But now we must be careful: $\rho = \dfrac{M_{sphere}}{\frac{4}{3}\pi R^3} \to \dfrac{M_{shell}}{4\pi R^2}$.

In other words, we need to use $\rho \to \dfrac{M_{shell}}{4\pi R^2}$.

$$I_{shell} = \frac{8}{3}\rho\pi R^4 \quad \Rightarrow \quad I_{shell} = \frac{8}{3}\frac{M_{shell}}{4\pi R^2}\pi R^4 = \frac{2}{3}M_{shell}R^2 \quad \Rightarrow \quad \boxed{I = \frac{2}{3}MR^2}$$

What is the kinetic energy of a rolling basketball? A basketball has air inside and serves as a good approximation for the shell. A rolling basketball rotates about its center as it rolls.

$$K = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}Mv^2 + \frac{1}{2}\frac{2}{3}MR^2\omega^2 = \frac{1}{2}Mv^2 + \frac{1}{2}\frac{2}{3}MR^2(\frac{v}{R})^2 = \frac{1}{2}\frac{5}{3}Mv^2 = \frac{5}{6}Mv^2$$

Courtesy forever_carrie_on, flickr, License: Attribution-NonCommercial-NoDerivs

2. About an Edge. The hanging disco ball below serves as an example of a spherical shell that can swing about its top edge, or at least close to the top. This fine ball is found in The Garden Brewery in Zagreb, Croatia. I believe it is the only brewery that displays a disco ball.

Courtesy Davor Čengija, flickr, Dedicated to the Public Domain
Disco Ball in The Garden Brewery, Zagreb, Croatia

We can use the parallel axis theorem to quickly find the answer.

$$I = I_{cm} + Md^2$$

$$I = \frac{2}{3}MR^2 + MR^2$$

$$\boxed{I = \frac{5}{3}MR^2}$$

L10. The Moment of Inertia for a Rectangle.

1. The Door. We will find the moment of inertia for a door using two methods.

Courtesy RyanIsHungry, flickr
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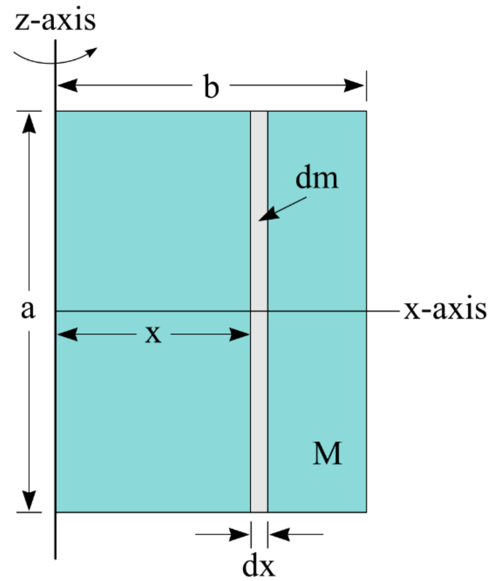
Method 1. Integration. The integration is almost identical to the thin rod we did earlier.



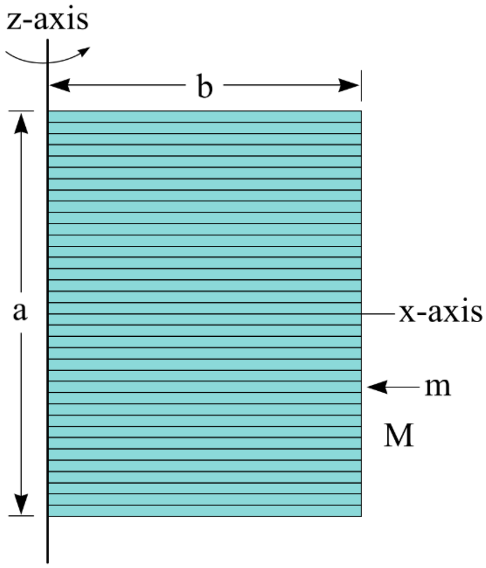
$$dI = x^2 dm \qquad dm = \sigma dA \qquad \sigma = \frac{M}{ab} \qquad dA = a dx$$

$$dI = x^2 dm = x^2 \sigma dA = x^2 \frac{M}{ab} dA = x^2 \frac{M}{ab} a(dx) = \frac{M}{b} x^2 dx$$

$$I = \frac{M}{b} \int_0^b x^2 dx = \frac{M}{b} \frac{x^3}{3} \bigg|_0^b = \frac{M}{b} \frac{b^3}{3}$$

$$\boxed{I = \frac{1}{3} M b^2}$$

This result is the same as that from the rod about its end.



Method 2. Using Rods. We build the door with n rods, where each rod has mass m.

$$I = \sum_{k=1}^{n} I_k$$

We know for a rod about its end: $I_{rod} = \frac{1}{3}mb^2$. We add all the moments of inertia for the rods.

$$I = \sum_{k=1}^{n} I_k = n\frac{1}{3}mb^2 = \frac{1}{3}(nm)b^2 = \frac{1}{3}Mb^2$$

The result is obtained super fast. It is the same as having one thick rod.

2. The Hope Chest Lid. The lid opens upward. The moment of inertia has the same formula.

Hand Crafted Hope Chest Courtesy DragonOak, flickr
[License: Attribution-NonCommercial-NoDerivs]

3. A Revolving Door Segment.

Courtesy Elliott Brown, flickr
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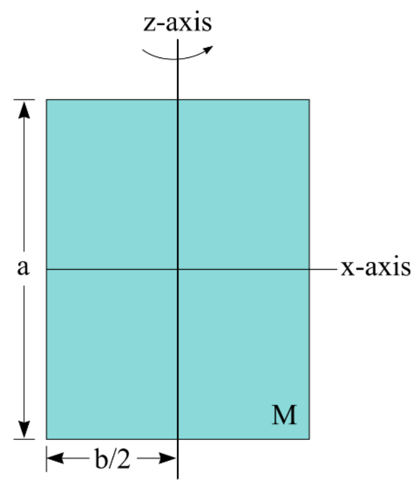


Method 1. We know when the axis is at the left edge

$$I = \frac{1}{3}Mb^2.$$

To get the moment of inertia about the center of mass, we use the parallel axis theorem

$$I = I_{cm} + Md^2,$$

where $d = \dfrac{b}{2}$ and $I = \dfrac{1}{3}Mb^2$.

$$I = I_{cm} + Md^2 \quad => \quad \frac{1}{3}Mb^2 = I_{cm} + M(\frac{b}{2})^2 \quad => \quad I_{cm} = \frac{1}{3}Mb^2 - M(\frac{b}{2})^2$$

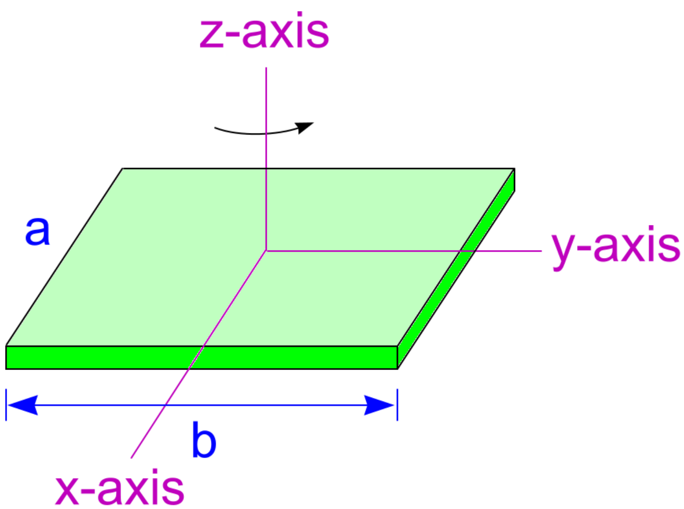
$$I_{cm} = (\frac{1}{3} - \frac{1}{4})Mb^2 \quad => \quad I_{cm} = (\frac{4-3}{12})Mb^2 \quad => \quad \boxed{I_{cm} = \frac{1}{12}Mb^2}$$

Method 2. Since the door and rod give the same results, we write down the result for the rod.

$$\boxed{I_{cm} = \frac{1}{12}Mb^2}$$

4. The Rotating Rectangular Table.

Rectangular Rotatable Table Courtesy Brian Evans, flickr
License: Attribution-NoDerivs 2.0 Generic



The perpendicular axis theorem comes to the rescue. We want the moment of inertia about the z-axis. From the theorem, we have

$$I_z = I_x + I_Y$$

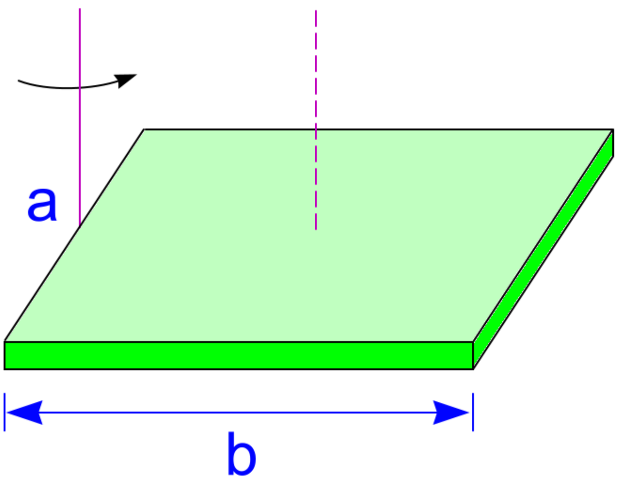
$$I_z = \frac{1}{12}Mb^2 + \frac{1}{12}Ma^2$$

$$\boxed{I_z = \frac{1}{12}Ma^2 + \frac{1}{12}Mb^2}$$

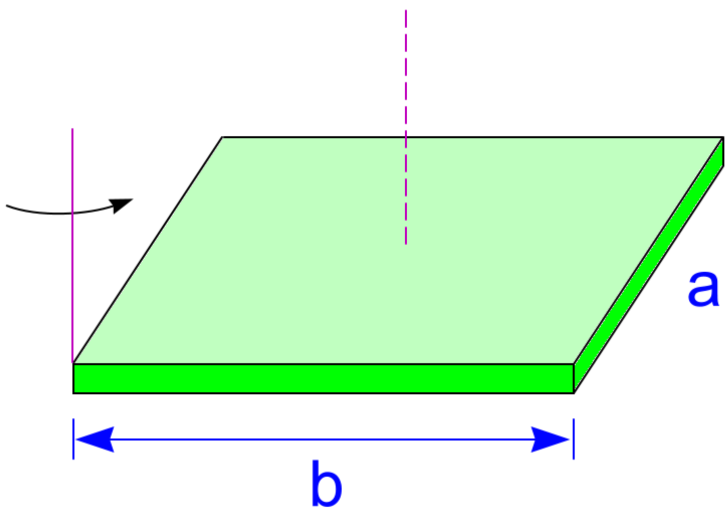
$$\boxed{I = \frac{1}{12} M (a^2 + b^2)}$$

The dimensions are correct and the symmetry in “a” and “b” is an additional check.

We can now easily find the moment of inertia about the middle of a side or even a corner.

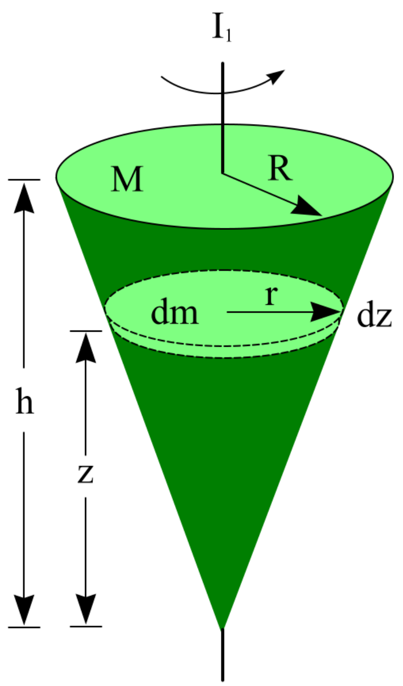


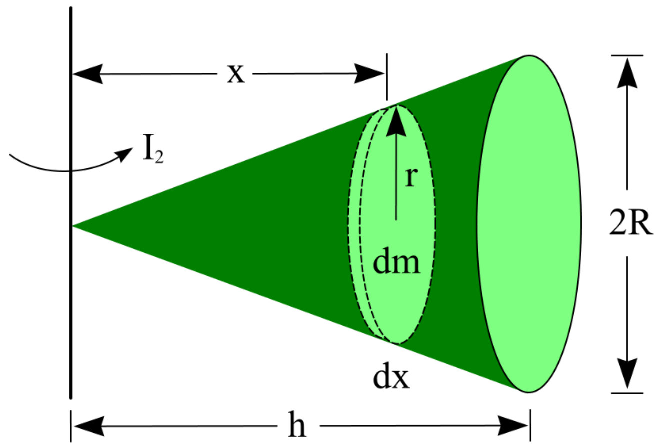
$$I_{mid-a-side} = \frac{1}{12} M (a^2 + b^2) + M (\frac{b}{2})^2 = \frac{1}{12} M (a^2 + b^2) + M \frac{b^2}{4} = \frac{1}{12} M (a^2 + 4b^2)$$



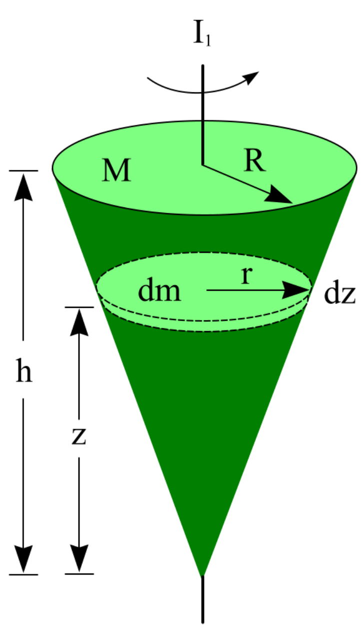
$$I_{corner} = \frac{1}{12} M (a^2 + b^2) + M \left[(\frac{a}{2})^2 + (\frac{b}{2})^2 \right] = \frac{1}{12} M (4a^2 + 4b^2) = \frac{1}{3} M (a^2 + b^2)$$

L11. The Cone. We will determine moments of inertia for a cone, depending on the axis. Below are the two cases we will analyze.





Case 1. Cone About Central Axis.



We will need the volume of the cone. This volume is not usually remembered. We can quickly calculate it and gain practice for our moments of inertia integration. From the figure,

$$dV = \pi r^2 dz\,.$$

We need to relate r to z. We do it by proportion.

$$\frac{r}{R} = \frac{z}{h} \quad => \quad r = \frac{z}{h} R$$

The differential volume element is then

$dV = \pi(\frac{R}{h} z)^2 dz$ and we can now find the volume.

$$V = \int_0^h \pi (\frac{R}{h} z)^2 dz$$

$$V = \pi \frac{R^2}{h^2} \int_0^h z^2 dz \quad => \quad V = \pi \frac{R^2}{h^2} \frac{z^3}{3}\Big|_0^h \quad => \quad V = \pi \frac{R^2}{h^2} \frac{h^3}{3}$$

$$\boxed{V = \frac{1}{3}\pi R^2 h}$$

Now we are warmed up for the moment of inertia integration. Start with $dI = \frac{1}{2} r^2 dm$ since the mass element is a disk and for a disk about its center of mass $I_{disk} = \frac{1}{2} M_{disk} R_{disk}^2$. Then,

$$dI = \frac{1}{2} r^2 dm \quad => \quad dI = \frac{1}{2} r^2 \rho dV \text{ where } \rho = \frac{M}{\frac{1}{3}\pi R^2 h} = \frac{3M}{\pi R^2 h} \text{ and } dV = \pi (\frac{R}{h} z)^2 dz .$$

$$dI = \frac{1}{2} r^2 \rho dV \quad => \quad dI = \frac{1}{2} r^2 \rho \pi (\frac{R}{h} z)^2 dz \quad => \quad dI = \frac{1}{2} r^2 \rho \pi \frac{R^2}{h^2} z^2 dz$$

Substitute $r = \frac{z}{h} R$.

$$dI = \frac{1}{2} r^2 \rho \pi \frac{R^2}{h^2} z^2 dz \quad => \quad dI = \frac{1}{2} (\frac{z}{h} R)^2 \rho \pi \frac{R^2}{h^2} z^2 dz$$

$$dI = \frac{1}{2} (\frac{z}{h} R)^2 \rho \pi \frac{R^2}{h^2} z^2 dz \quad => \quad dI = \frac{1}{2} \rho \pi \frac{R^4}{h^4} z^4 dz$$

$$I = \frac{1}{2} \int_0^h \rho \pi \frac{R^4}{h^4} z^4 dz$$

$$I = \frac{1}{2} \rho \pi \frac{R^4}{h^4} \int_0^h z^4 dz$$

$$I = \frac{1}{2} \rho \pi \frac{R^4}{h^4} \frac{z^5}{5}\Big|_0^h$$

$$I = \frac{1}{2}\rho\pi\frac{R^4}{h^4}\frac{h^5}{5}$$

$$I = \rho\pi R^4\frac{h}{10}$$

$$\rho = \frac{3M}{\pi R^2 h} \quad => \quad I = \frac{3M}{\pi R^2 h}\pi R^4\frac{h}{10} \quad => \quad I = \frac{3M}{h}R^2\frac{h}{10} \quad => \quad I = 3MR^2\frac{1}{10}$$

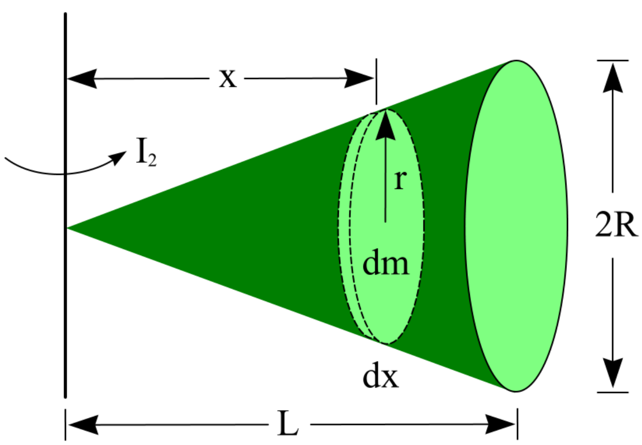
$$\boxed{I = \frac{3}{10}MR^2}$$

Note that the h dropped out!

Since we are doing another case, let’s remind ourselves with did case 1 by including a subscript.

$$\boxed{I_{central} = \frac{3}{10}MR^2}$$

Case 2. Cone with Axis Through Pointed End.



We start by moving the disk element shown in the figure so that this element at the axis. The disk has radius r and mass dm and it is standing on its edge at x = 0:

$$dI_{x=0} = \frac{1}{4}(dm)r^2 .$$

Do you remember how we obtained the above result from the perpendicular axis theorem in order to get the 1/4?

Then we shift it to the right using the parallel axis theorem.

$$dI = \frac{1}{4}(dm)r^2 + (dm)x^2$$

The next step is work with dm: $dm = \rho\pi r^2 dx$ with $\rho = \frac{3M}{\pi R^2 h}$ from the last problem.

$$dI = \frac{1}{4}(dm)r^2 + (dm)x^2 \quad => \quad dI = (\frac{1}{4}r^2 + x^2)dm \quad => \quad dI = (\frac{1}{4}r^2 + x^2)\rho\pi r^2 dx$$

$$dI = (\frac{1}{4}r^2 + x^2)\rho\pi r^2 dx \quad => \quad dI = \rho\pi(\frac{1}{4}r^4 + x^2 r^2)dx$$

We need to get r in terms of x. We do it by proportion similar to what we did in the last problem.

$$r = \frac{x}{h}R$$

Proceeding with this substitution,

$$dI = \rho\pi(\frac{1}{4}r^4 + x^2 r^2)dx \quad => \quad dI = \rho\pi(\frac{1}{4}\frac{x^4}{h^4}R^4 + x^2\frac{x^2}{h^2}R^2)dx$$

$$dI = \rho\pi(\frac{1}{4}\frac{x^4}{h^4}R^4 + \frac{x^4}{h^2}R^2)dx \quad => \quad dI = \rho\pi\frac{R^2}{h^2}(\frac{1}{4}\frac{x^4}{h^2}R^2 + x^4)dx$$

$$dI = \rho\pi\frac{R^2}{h^2}(\frac{1}{4}\frac{R^2}{h^2} + 1)x^4 dx$$

We now integrate from x = 0 to x = h.

$$I = \rho\pi\frac{R^2}{h^2}(\frac{1}{4}\frac{R^2}{h^2} + 1)\int_0^h x^4 dx$$

$$I = \rho\pi\frac{R^2}{h^2}(\frac{1}{4}\frac{R^2}{h^2} + 1)\frac{x^5}{5}\Big|_0^h$$

$$I = \rho\pi\frac{R^2}{h^2}(\frac{1}{4}\frac{R^2}{h^2} + 1)\frac{h^5}{5}$$

$$I = \rho\pi R^2(\frac{1}{4}\frac{R^2}{h^2} + 1)\frac{h^3}{5}$$

Now substitute in for $\rho = \frac{3M}{\pi R^2 h}$.

$$I = \frac{3M}{\pi R^2 h}\pi R^2(\frac{1}{4}\frac{R^2}{h^2} + 1)\frac{h^3}{5}$$

$$I = \frac{3M}{h}(\frac{1}{4}\frac{R^2}{h^2}+1)\frac{h^3}{5}$$

$$I = 3M(\frac{1}{4}\frac{R^2}{h^2}+1)\frac{h^2}{5}$$

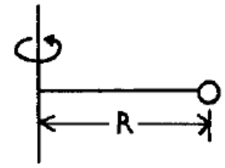
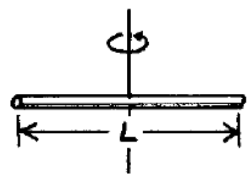
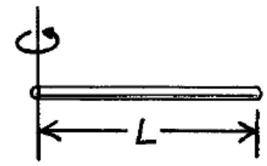
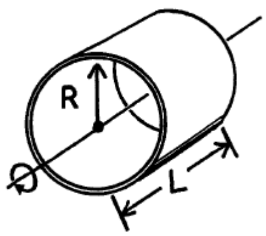
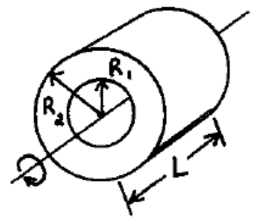
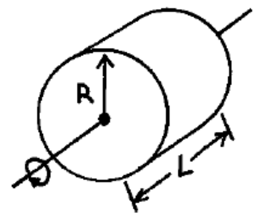
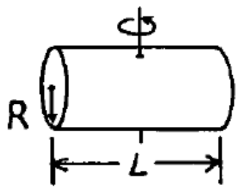
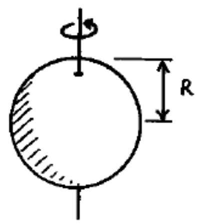
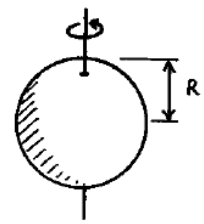
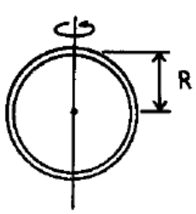
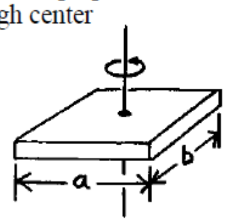
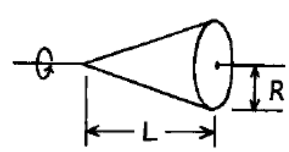
$$I = 3M(\frac{1}{4}R^2+h^2)\frac{1}{5}$$

$$I = \frac{3}{4}M(R^2+4h^2)\frac{1}{5}$$

$$\boxed{I_{\text{point}\perp} = \frac{3}{20}M(R^2+4h^2)}$$

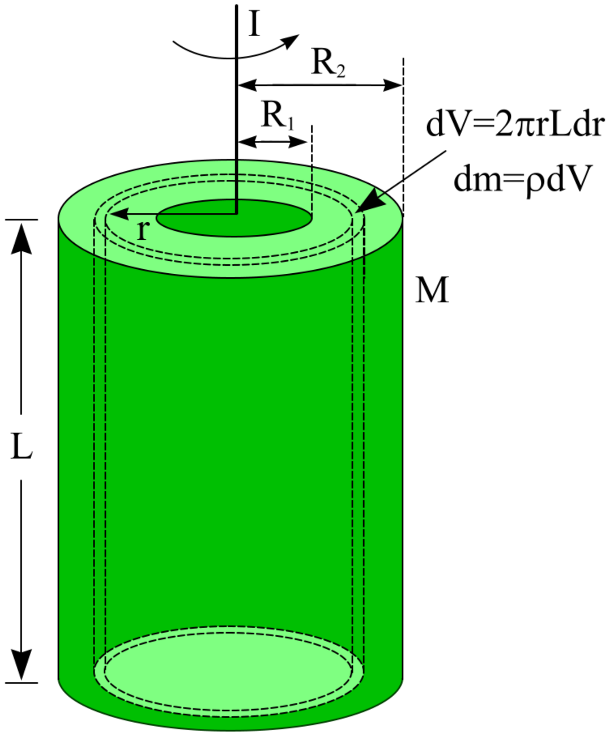
This moment of inertial is the moment of inertial of a cone about an axis perpendicular to its central axis and where its apex touches the axis of rotation.

L12. Table of Moments of Inertia.

Point mass at a radius R  $I = MR^2$	Thin rod about axis through center perpendicular to length  $I = \frac{1}{12}ML^2$	Thin rod about axis through end perpendicular to length  $I = \frac{1}{3}ML^2$
Thin-walled cylinder about central axis  $I = MR^2$	Thick-walled cylinder about central axis  $I = \frac{1}{2}M(R_1^2 + R_2^2)$	Solid cylinder about central axis  $I = \frac{1}{2}MR^2$
Solid cylinder about central diameter  $I = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$	Solid sphere about center  $I = \frac{2}{5}MR^2$	Thin hollow sphere about center  $I = \frac{2}{3}MR^2$
Thin ring about diameter  $I = \frac{1}{2}MR^2$	Slab about perpendicular axis through center  $I = \frac{1}{12}M(a^2 + b^2)$	Cone about central axis  $I = \frac{3}{10}MR^2$

Note: All formulas shown assume objects of uniform mass density.

The only entry we have not derived is the thick-walled cylinder. Let's derive that formula now.



$$dI = r^2 dm$$

$$dI = r^2 \rho dV$$

$$dI = r^2 \rho [2\pi r L dr] \qquad \rho = \frac{M}{\pi(R_2^2 - R_1^2)L}$$

$$I = \int_{R_1}^{R_2} r^2 \rho [2\pi r L dr]$$

$$I = 2\pi\rho L \int_{R_1}^{R_2} r^3 dr$$

$$I = 2\pi\rho L \left. \frac{r^4}{4} \right|_{R_1}^{R_2}$$

$$I = 2\pi\rho L \left(\frac{R_2^4}{4} - \frac{R_1^4}{4}\right)$$

$$I = \frac{1}{2}\pi\rho L(R_2^4 - R_1^4)$$

$$I = \frac{1}{2}\pi \frac{M}{\pi(R_2^2 - R_1^2)L} L(R_2^4 - R_1^4)$$

$$I = \frac{1}{2}M \frac{(R_2^4 - R_1^4)}{(R_2^2 - R_1^2)}$$

$$I = \frac{1}{2}M \frac{(R_2^2 + R_1^2)(R_2^2 - R_1^2)}{(R_2^2 - R_1^2)}$$

$$I = \frac{1}{2}M(R_2^2 + R_1^2)$$

$$\boxed{I = \frac{1}{2}M(R_1^2 + R_2^2)}$$