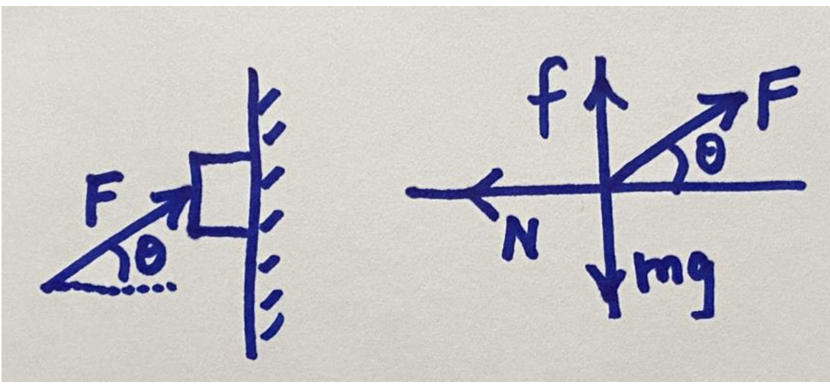
Physics I with Calculus, Prof. Ruiz (Doc), UNC-Asheville (1978-2021), DoctorPhys on YouTube
Chapter G. Work. Prerequisite: Calculus I. Corequisite: Calculus II.

Before starting our new topic of work, let's review friction.

G1. Static Friction: Holding a Block Up Against a Wall. This problem was inspired by an analysis made by Rhett Allain, Associate Professor of Physics at Southeastern Louisiana University (2002). He calls it "The Surprisingly Cool Physics of Pushing a Block Against a Wall." Below is my approach to this neat problem. Remember that with the "Method of Science" scientists have their own unique ways in approaching problems, but in the end, they should get the same results. Science is reproducible. So we will obtain the same results in our analysis.



See the left figure for our first two steps. We want the minimum force F to keep the block from falling. Therefore, the frictional force will point upward.

(i) Sketch. A sketch is better than nothing. I did a free hand sketch.

(ii) Force Diagram. Once again, I draw with free hand. Your teacher will usually set guidelines on homework. For introductory physics, a neat figure drawn by hand typically suffices. Never cross out on homework. Next comes Newton's Second Law.

(iii) Newton's Second Law. The acceleration is zero. We have static equilibrium. Choose "+" in the usual way for a horizontal (x) and vertical (y) coordinate system.

$$\sum F_x = F\cos\theta - N = 0$$

$$\sum F_y = F\sin\theta + f - mg = 0$$

We are now ready for our friction equation, which I like to call an auxiliary equation. Since we want the condition where the mass is just above to slide down, we have

$$f = \mu_s N .$$

Note that we are not trying to slide the block up the incline. Instead, we are pushing with the minimum force needed to keep the block from sliding down the wall. The result is an upward frictional force. Our three equations are listed again below. We are ready for the last phase.

(iv) Solve.

$$F\cos\theta - N = 0$$

$$F\sin\theta + f - mg = 0$$

$$f = \mu_s N$$

As mentioned in the last chapter, I like to start with the bottom equation and work up. So I look at $f = \mu_s N$ and see I need N. Then I note that the simplest expression for N is found in the first equation, where I obtain $N = F\cos\theta$. Therefore, we get

$$f = \mu_s N = \mu_s F\cos\theta\,.$$

Now we substitute this expression for the friction in the middle equation $F\sin\theta + f - mg = 0$.

$$F\sin\theta + \mu_s F\cos\theta - mg = 0$$

We want to solve for the force F.

$$F\sin\theta + \mu_s F\cos\theta - mg = 0 \quad => \quad F(\sin\theta + \mu_s\cos\theta) - mg = 0$$

$$F(\sin\theta + \mu_s\cos\theta) = mg$$

$$\boxed{F(\theta) = \frac{mg}{\sin\theta + \mu_s\cos\theta}}$$

A quick check is taking $\theta = 90°$. The force is then $F(90°) = \dfrac{mg}{\sin 90° + \mu_s\cos 90°} = \dfrac{mg}{1 + \mu_s \cdot 0} = mg$.

The applied force is directly upward and we are holding the mass up against gravity. We would like to find the best angle θ to achieve the minimum force needed to keep the mass from moving. We can do the standard max-min problem from calculus by setting the derivative equal to zero.

$$\frac{dF(\theta)}{d\theta} = 0\,.$$

But this derivative looks messy. For the minimum $F(\theta)$ however, the denominator will be a maximum. So we can do the max-min problem on the denominator.

$$\frac{d}{d\theta}(\sin\theta + \mu_s\cos\theta) = 0$$

$$\cos\theta - \mu_s \sin\theta = 0 \quad => \quad \cos\theta = \mu_s \sin\theta \quad => \quad 1 = \mu_s \frac{\sin\theta}{\cos\theta}$$

$$1 = \mu_s \tan\theta$$

$$\boxed{\tan\theta_0 = \frac{1}{\mu_s}}$$

We include the subscript "0" to remind us that this angle is special, the angle that gives the minimum force.

$$\boxed{F_{\min} = F(\theta_0) = \frac{mg}{\sin\theta_0 + \mu_s \cos\theta_0}}$$

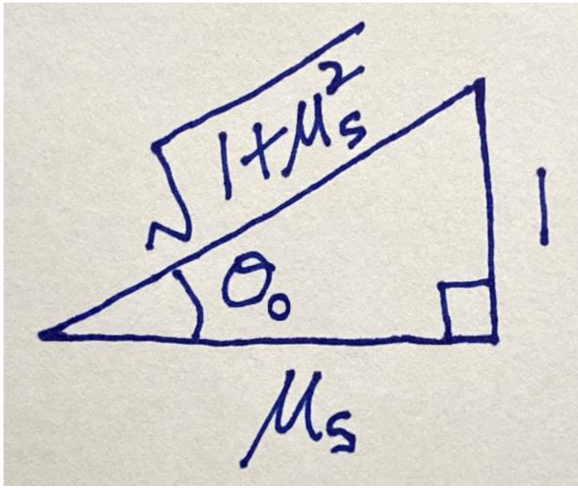
The table below gives the best angle θ_0 for some values of μ_s and the associated force $F(\theta_0)$.

μ_s	$\theta_0 = \tan^{-1}\frac{1}{\mu_s}$	$F(\theta_0)$ in N	$F(\theta_0)/(mg)$
0.00	90.0°	9.80	1.000
0.10	84.3°	9.75	0.995
0.20	78.7°	9.61	0.981
0.30	73.3°	9.39	0.958
0.40	68.2°	9.10	0.928
0.50	63.4°	8.77	0.894
0.60	59.0°	8.40	0.857
0.70	55.0°	8.03	0.819
0.80	51.3°	7.65	0.781
0.90	48.0°	7.28	0.743
1.00	45.0°	6.93	0.707
1.10	42.3°	6.59	0.673
1.20	39.8°	6.27	0.640

As $\mu_s \to 0$ we have $1/\mu_s \to \infty$, $\theta_0 = \tan^{-1}(1/\mu_s) \to \pi/2 = 90°$, and $F_{\min} \to mg$. The best deal is to simply exert an upward force to hold up the mass against gravity. With 90° you do not waste any effort with a force component towards the wall since pushing against the wall does not help you if there is no friction.

As μ_s increases, θ_0 decreases. For $\mu_s \to \infty$ we can consider the block glued to the wall. Then $1/\mu_s \to 0$, the angle $\theta_0 = \tan^{-1}(1/\mu_s) \to 0°$ and $F_{\min} \to 0$. If there is a finite and large μ_s, you gently push towards the wall. Your angle will be $\theta_0 \approx 0°$ and your force will be small, $F_{\min} \approx 0$.

We can express the minimum force solely in terms of the coefficient of static friction μ_s. I will use a trick that I learned from a physics friend long ago. The trick is to construct a right triangle so that $\tan\theta_0 = \dfrac{1}{\mu_s}$.



From the figure at the left we can get $\sin\theta_0$ and $\cos\theta_0$.

$$\sin\theta_0 = \frac{1}{\sqrt{1+\mu_s^2}} \qquad \cos\theta_0 = \frac{\mu_s}{\sqrt{1+\mu_s^2}}$$

$$F_{\min} = \frac{mg}{\sin\theta_0 + \mu_s\cos\theta_0}$$

$$F_{\min} = \frac{mg}{\dfrac{1}{\sqrt{1+\mu_s^2}} + \mu_s\dfrac{\mu_s}{\sqrt{1+\mu_s^2}}}$$

Multiply top and bottom by $\sqrt{1+\mu_s^2}$.

$$F_{\min} = \frac{\sqrt{1+\mu_s^2}}{1+\mu_s^2}mg$$

$$\boxed{F_{\min} = \frac{mg}{\sqrt{1+\mu_s^2}}} \qquad \boxed{\tan\theta_0 = \frac{1}{\mu_s}}$$

Is this answer reasonable? Consider approaching no friction. Then $\mu_s \to 0$ and

$$\lim_{\mu_k\to 0} F_{\min} = \lim_{\mu_k\to 0}\frac{mg}{\sqrt{1+\mu_s^2}} = mg\ ,$$

with the angle becoming

$$\lim_{\mu_s\to 0}\theta_0 = \lim_{\mu_s\to 0}\tan^{-1}\frac{1}{\mu_s} = \lim_{x\to\infty}\tan^{-1}x = 90^\circ .$$

We are holding the mass straight up and not wasting any force pushing into the wall. Let's again consider the mass nailed to the wall, i.e., $\mu_s \to \infty$.

$$\lim_{\mu_k\to\infty} F_{\min} = \lim_{\mu_k\to\infty}\frac{mg}{\sqrt{1+\mu_s^2}} = 0$$

As we found earlier, no force is needed. The angle is really irrelevant, but the formula gives

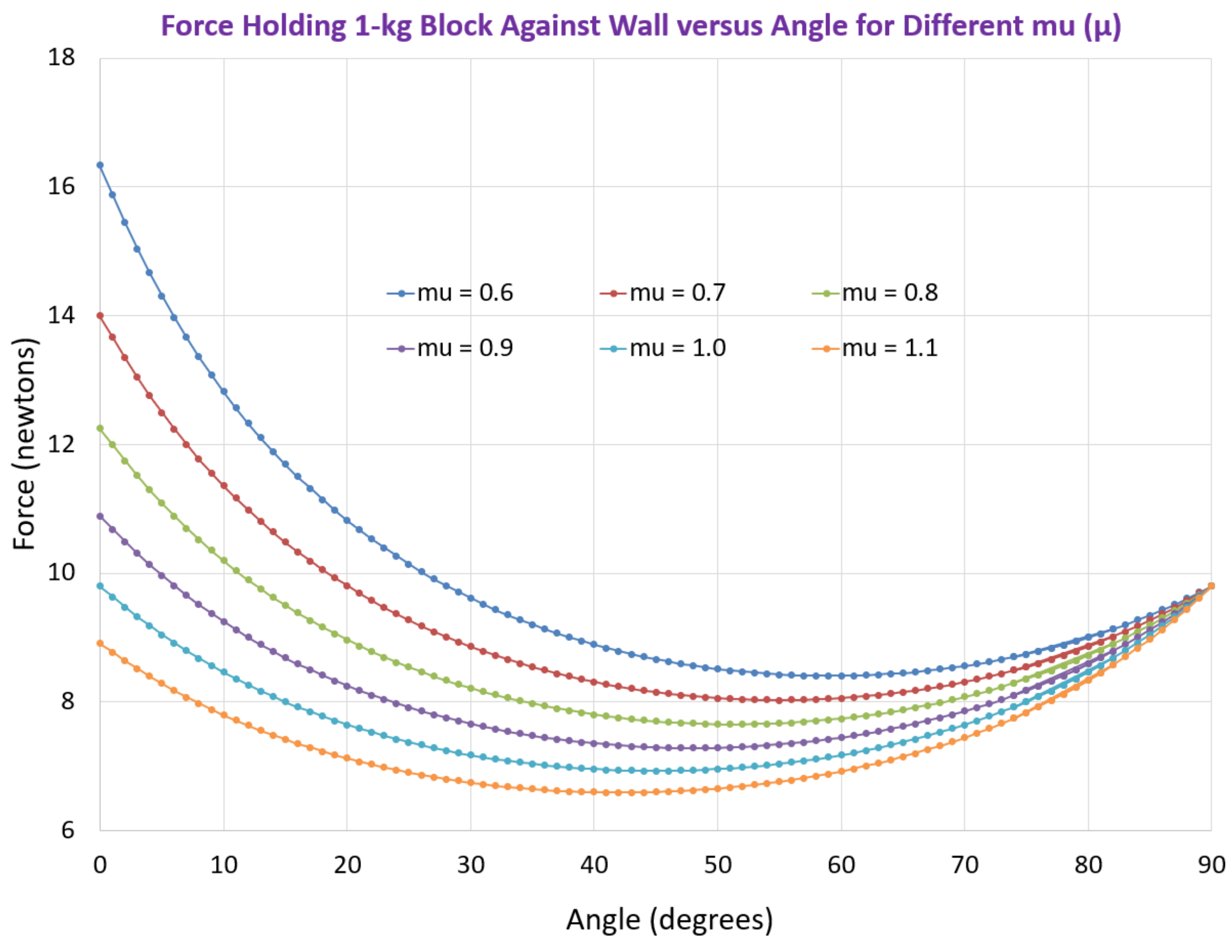
$$\lim_{\mu_s \to \infty} \theta_0 = \lim_{\mu_s \to \infty} \tan^{-1} \frac{1}{\mu_s} = \lim_{x \to 0} \tan^{-1} x = 0° .$$

However, if the mass was a little loose, you would push gently straight towards the wall.

Professor Allain gives nice plots of the force as a function for all angles with different values of μ_s using Python.

$$F(\theta) = \frac{mg}{\sin\theta + \mu_s \cos\theta}$$

I used a spreadsheet to obtain a few plots. Eyeball the minima and compare your estimates with the above table.

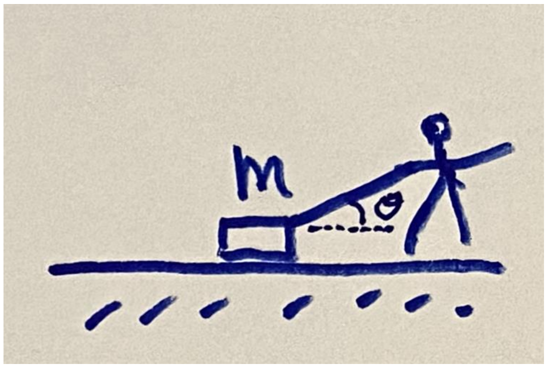


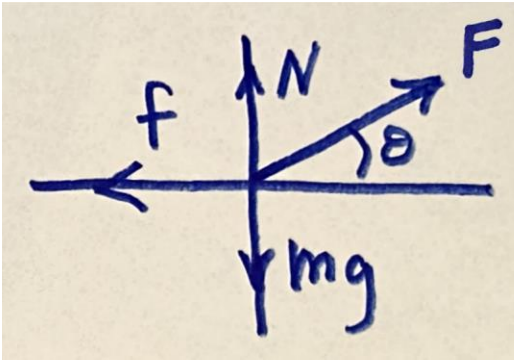
Observe how all the results agree at 90°. You are then holding the block up by its weight.

$$\lim_{\theta \to 90°} F(\theta) = \lim_{\theta \to 90°} \frac{mg}{\sin\theta + \mu_s \cos\theta} = \frac{mg}{\sin 90° + \mu_s \cos 90°} = \frac{mg}{1 + \mu_s \cdot 0} = mg$$

G2. Kinetic Friction: Dragging a Sled Across the Snow. Constant pulling with constant speed.

Colin and Mom, Courtesy Patrick W., flickr. [License: Attribution 2.0 Generic.]()
Photo taken March 5, 2006. Slight contrast enhancement on the photo.





Choose directions as usual with an x-y coordinate system. Right is then positive "x" and up is positive "y".

$$\sum F_x = F\cos\theta - f = ma = 0$$

$$\sum F_y = F\sin\theta + N - mg = 0$$

We have three equations when we add the friction equation. The three equations are below.

$$F\cos\theta - f = 0$$

$$F\sin\theta + N - mg = 0$$

$$f = \mu_k N$$

Let's solve for the force F. Start with $f = \mu_k N$ and use the second equation to get the normal force $N = mg - F\sin\theta$ for substitution into our friction equation.

$$f = \mu_k N = \mu_k(mg - F\sin\theta)$$

Then place this force into the top equation.

$$F\cos\theta - \mu_k(mg - F\sin\theta) = 0$$

Continue with the solution for F.

$$F(\cos\theta + \mu_k F\sin\theta) - \mu_k mg = 0$$

$$F(\cos\theta + \mu_k \sin\theta) = \mu_k mg$$

$$\boxed{F(\theta) = \frac{\mu_k mg}{\cos\theta + \mu_k \sin\theta}}$$

What is the best angle to apply the force F? Pulling up some eases the frictional force since the normal force is reduced. We can set the derivative to zero: $\dfrac{dF(\theta)}{d\theta} = 0$. But as before, we rather find the extremum for the denominator. A max for the denominator will give a min for the force.

$$\frac{d(\cos\theta + \mu_k \sin\theta)}{d\theta} = 0$$

$$-\sin\theta + \mu_k \cos\theta = 0$$

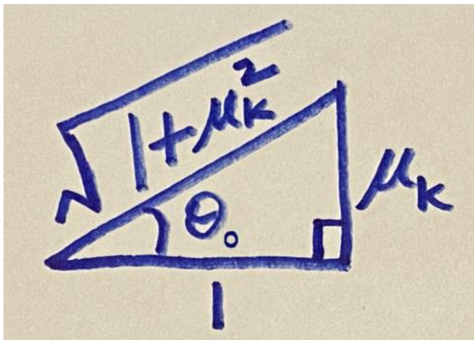
$$\mu_k \cos\theta = \sin\theta$$

$$\boxed{\tan\theta_0 = \mu_k}$$

We include the subscript to signal that we have a special angle since the generic variable θ appears in the force equation. The above special angle will give the case for minimal force.

$$F_{\min} = F(\theta_0) = \frac{\mu_k mg}{\cos\theta_0 + \mu_k \sin\theta_0}$$

We use the triangle trick to get the trig functions in terms of μ_k. We construct a right triangle so that we have an angle with tangent $\tan\theta_0 = \mu_k$.



$$\cos\theta_0 = \frac{1}{\sqrt{1+\mu_k^2}}$$

$$\sin\theta_0 = \frac{\mu_k}{\sqrt{1+\mu_k^2}}$$

$$F_{\min} = \frac{\mu_k mg}{\frac{1}{\sqrt{1+\mu_k^2}} + \mu_k \frac{\mu_k}{\sqrt{1+\mu_k^2}}}$$

Multiply top and bottom by $\sqrt{1+\mu_k^2}$.

$$F_{\min} = \frac{\mu_k \sqrt{1+\mu_k^2}\, mg}{1+\mu_k^2}$$

$$\boxed{F_{\min} = \frac{\mu_k mg}{\sqrt{1+\mu_k^2}}} \qquad \boxed{\tan\theta_0 = \mu_k}$$

Compare these result to the problem in the previous section.

Colin and Mom, Courtesy Patrick W., flickr. [License: Attribution 2.0 Generic.](#)
Photo taken March 5, 2006. Slight contrast enhancement on photo.

Take a sled with a child on it to be 150.0 N (33.7 lb) and the coefficient of kinetic friction for this sled on snow to be 0.1. What is the best angle to pull at and what is the force you need?

$$\theta_0 = \tan^{-1}\mu_k = \tan^{-1} 0.1 = 5.7°$$

$$F_{\min} = \frac{\mu_k mg}{\sqrt{1+\mu_k^2}} = \frac{0.1\cdot(150\text{ N})}{\sqrt{1+0.1^2}} = \frac{15}{\sqrt{1.01}}\text{ N} = 14.9\text{ N} = 15\text{ N}$$

But mom finds the small angle 5.7° too awkward since she would have to stoop down too much. So she uses an angle like we see in the above photo, an angle more like 40°. To find the force now, we go to the general force equation

$$F(\theta) = \frac{\mu_k mg}{\cos\theta + \mu_k \sin\theta}$$

and insert the values $\mu_k = 0.1$, $mg = 150.0\text{ N}$, and $\theta = 40°$.

$$F = \frac{\mu_k mg}{\cos\theta + \mu_k \sin\theta} = \frac{0.1(150\text{ N})}{\cos 40° + 0.1\cdot\sin 40°} = \frac{15}{0.76604 + 0.06428}\text{ N} = 18.0653\text{ N}$$

$$F = 18\text{ N}$$

This value is not much greater than the minimum case of $F_{\min} = 15\text{ N}$ and it is a lot easier to pull at this angle.

G3. Work. We will define a new physical quantity called work in this section. We will arrive at this physical quantity by considering hiring workers to push merchandise in boxes across a warehouse floor to a loading area where a large truck awaits.

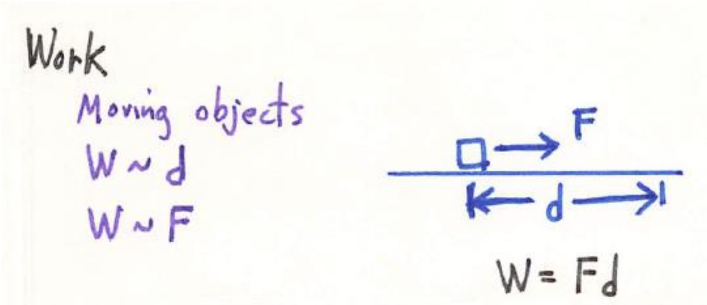
Push, Courtesy ditch Mingo, flickr
[License: Attribution-NonCommercial-NoDerivs 2.0 Generic (CC BY-NC-ND 2.0)]

At the left is a possible worker. By the way, our definition will not matter if the merchandise is on wheels or not. Suppose we want to pay our employees for the actual work they do? What is this work?

Two characteristics emerge:

1. The Force F. Workers pushing more massive objects should be paid more.
2. The Distance d. Workers pushing greater distances should be paid more.

Some items in the warehouse are farther from the loading dock and so there is greater distance d to cover for these. Taking both of these features into consideration, it makes sense to define work as the product of the force and the distance.



Our new physical quantity is work. The following definition applies to one dimension where the force is in the direction of the motion – the most simple pushing of objects.

$$W = Fd$$

The dimensions are newtons times meters. We name this combination after the scientist James Joule (1818-1889). Joule was an English physicist that related mechanical work to heat energy. We will study heat later. Spelling out the units, the definition of the joule is

$$\text{joule} = \text{newton} \cdot \text{meter} \,,$$

where we use lowercase "j" just as we do for newton. For the symbol, i.e., the abbreviated form, we have

$$\mathrm{J} = \mathrm{N} \cdot \mathrm{m} \,,$$

where uppercase is used for the joule and newton. If you want to express J in terms of the three fundamental quantities of length, time, and mass, we can use $F = ma$ to first express N.

$$\mathrm{N} = \mathrm{kg} \cdot \frac{\mathrm{m}}{\mathrm{s}^2} \,.$$

Then the joule is

$$\mathrm{J} = \mathrm{N} \cdot \mathrm{m} = \mathrm{kg} \cdot \frac{\mathrm{m}}{\mathrm{s}^2} \cdot \mathrm{m} \,,$$

$$\mathrm{J} = \mathrm{kg} \cdot \frac{\mathrm{m}^2}{\mathrm{s}^2} \,.$$

In the cgs system we have dynes (force) times centimeters (distance), which is defined as the erg.

$$\mathrm{erg} = \mathrm{dyne} \cdot \mathrm{centimeter}$$

In the British system, we have pounds times feet, which is often referred to as

$$\mathrm{lb} \cdot \mathrm{ft} \text{ or } \mathrm{ft} \cdot \mathrm{lb} .$$

With the notation of dimensional analysis, we can write

$$[\mathrm{distance}] = L \text{, } [\mathrm{time}] = T \text{, and } [\mathrm{mass}] = M .$$

Then, the dimensions of work can be expressed as

$$[W] = [F][d] = [ma][d] = [m][a][d] = M \cdot \frac{L}{T^2} \cdot L = M \cdot \frac{L^2}{T^2} ,$$

$$[W] = ML^2T^{-2} .$$

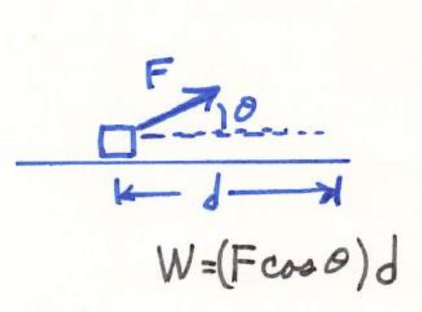
In an ideal scenario, we can pay our workers by the work they accomplish in joules rather than by the hour. Consider again our definition of work.

$$W = F \cdot d$$

There are two immediate ways to see when no work is done.

Case 1. Walking like a zombie pushing nothing: $F = 0$ leading to $W = F \cdot d = 0 \cdot d = 0$.
Case 2. Pushing an object stuck up against a wall: $d = 0$ leading to $W = F \cdot d = F \cdot 0 = 0$.

The definition of work when the force is applied at an angle involves taking the component of the force along the direction of the motion.



The definition for work is then given by

$$W = Fd\cos\theta .$$

The force and distance are vectors, i.e., they have magnitudes and directions. However, the work is a scalar. There is no direction intrinsic to work itself. We make the following definition for combining two vectors to form a scalar. It is called the *vector dot product* or simply *dot product* and we write it as

$$\boxed{W = \vec{F} \cdot \vec{d} \equiv Fd\cos\theta}$$

Our ideal model for work breaks down somewhat since you may be able to push boxes more easily by pushing up a little to reduce the frictional force between the bottom of the crate and the ground. You would get no credit for this clever technique because the strict physics definition gives maximum work credit when $\theta = 0°$ and the greater force is used.

G4. Kinetic Energy. We introduce another new physical quantity in this section. This new quantity follows naturally by applying the definition of work with Newton's Second Law of Motion. We will provide three methods, but I like the second one the best for this course.

Method 1. Imagine an object in outer space at rest. Apply a constant force F and the object starts to accelerate according to Newton's Second Law. The velocity increases with constant acceleration. Then, at some point no longer apply the force. The mass will then continue at its final speed v achieved at the end of the acceleration phase. Let the force F be applied moving the object a distance equal to x. The work is

$$W = Fx .$$

We then introduce Newton's Law, $F = ma$. During the constant acceleration phase, the object accelerates according to the equation $a = \frac{v - v_0}{t} = \frac{v}{t}$ since it started from rest. Also, the distance traveled is given by $x = \frac{1}{2}(v_0 + v)t = \frac{1}{2}vt$ since $v_0 = 0$. Combining these relations, we find

$$W = Fx = max = m \cdot \frac{v}{t} \cdot \frac{1}{2}vt = \frac{1}{2}mv^2$$

The work that we did has transformed into energy in the motion. The moving mass can crash into something later doing work on something else. This energy of motion is called *kinetic energy* and is represented by K.

$$\boxed{K = \frac{1}{2}mv^2}$$

Method 2. Let's redo the one-dimensional derivation where the mass is initially traveling at speed v_0. Then, we can use the kinematic equations

$$v = v_0 + at ,$$

$$x = x_0 + \frac{1}{2}(v + v_0)t .$$

We will define our reference position so $x_0 = 0$. Then we have the following equations:

$$a = \frac{v - v_0}{t} \quad \text{and} \quad x = \frac{1}{2}(v + v_0)t .$$

$$W = Fx = (ma)x = (m \cdot \frac{v - v_0}{t}) \cdot \frac{1}{2}(v + v_0)t = \frac{1}{2}m(v - v_0)(v + v_0) = \frac{1}{2}m(v^2 - v_0^2)$$

$$\boxed{W = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2}$$

The work done is equal to the change in kinetic energy! The physical quantity

$$\boxed{K = \frac{1}{2}mv^2}$$

again emerges naturally!

Method 3. Integral Calculus. A corequisite for this course is taking Calculus II concurrently. Therefore, you might not have gotten far enough in your course to understand this method. If not, no problem. The main method I find most suitable for this course is Method 2, which we just finished. It uses the kinematic formulas we developed earlier. However, the calculus method has some increased powers as we do not have to assume a constant force and associated constant acceleration. Our kinematic equations will not help here since the force is not constant and therefore our kinematic equations do not apply.

Now we take the force as a function of x, i.e., $F = F(x)$. Watch what happens with calculus.

$$W = \int_{x_0}^{x} F(x)dx = \int_{x_0}^{x} madx = \int_{x_0}^{x} m\frac{dv}{dt}dx = \int_{o}^{t} m\frac{dv}{dt}\frac{dx}{dt}dt = \int_{o}^{t} mv\frac{dv}{dt}dt = m\int_{v_0}^{v} vdv = m\left.\frac{v^2}{2}\right|_{v_0}^{v}$$

$$\boxed{W = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2}$$

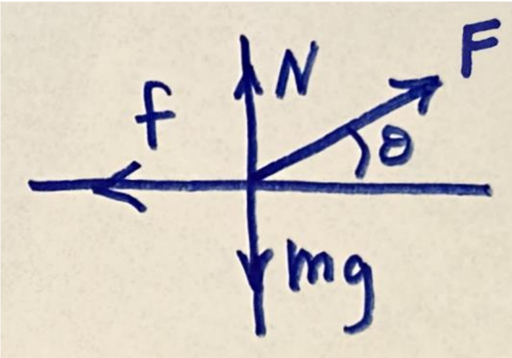
The derivation included a little “chain-rule” action on the derivatives.

If you did not get far enough in Calculus II, just skip the above method until you do.

G5. Work and Forces on an Object with No Acceleration. It is instructive to analyze an object moving at constant speed and calculate the work on it due to each individual force acting on it.

Colin and Mom, Courtesy Patrick W., flickr. License: Attribution 2.0 Generic.
Photo taken March 5, 2006. Slight contrast enhancement on photo.

We can consider pulling a sled at constant speed for our analysis. Since there is no gain in velocity, the net force must be doing no work. Let's consider each force separately first. There are four forces to check out. The distance traveled is d.



1. The Pulling Force F. The work done is

$$W_F = \vec{F} \cdot \vec{d} = (F \cos\theta)d$$

2. The Normal Force N.

$$W_N = \vec{N} \cdot \vec{d} = N(\cos 90°)d = 0$$

3. The Frictional Force f.

$$W_f = \vec{f} \cdot \vec{d} = (f \cos 180°)d = -fd$$

4. The Force Due to Gravity.

$$W_g = m\vec{g} \cdot \vec{d} = mg\cos(-90°)d = 0$$

We add these up to get the total work.

$$W_{Total} = W_F + W_N + W_f + W_g = (F\cos\theta)d + 0 - fd + 0$$

$$W_{Total} = (F\cos\theta)d - fd$$

$$W_{Total} = (F\cos\theta - f)d$$

But from the force diagram: $F\cos\theta = f$ for no acceleration. Therefore,

$$W_{Total} = 0$$

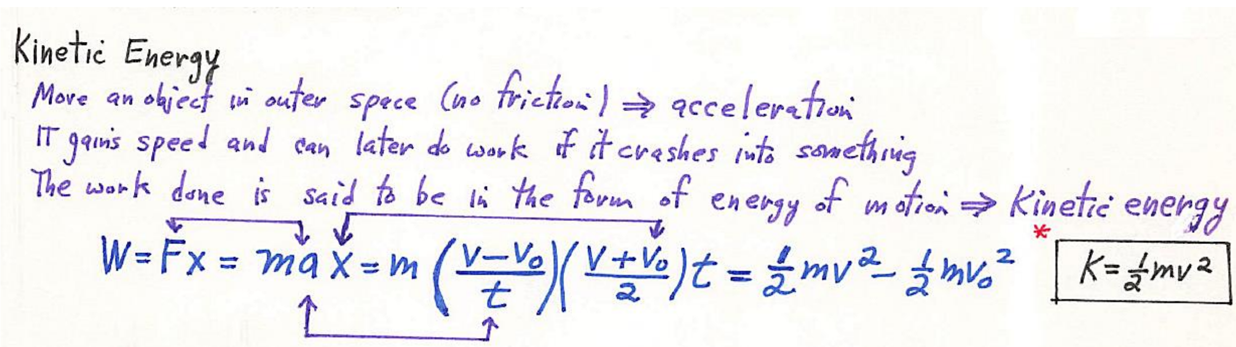
We can arrive at this zero another way – by first finding the net force. For constant velocity, the acceleration is zero. The net is therefore

$$\vec{F}_{net} = 0 .$$

Calculating the work due to the net force is then

$$W_{net} = \vec{F}_{net} \cdot \vec{d} = \vec{0} \cdot \vec{d} = 0 .$$

G6. The Work Energy Theorem. The Work-Energy Theorem follows from our last section, which we summarize below.


Kinetic Energy
Move an object in outer space (no friction) ⇒ acceleration
It gains speed and can later do work if it crashes into something
The work done is said to be in the form of energy of motion ⇒ Kinetic energy

$$W = Fx = max = m\left(\frac{v-v_0}{t}\right)\left(\frac{v+v_0}{2}\right)t = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 \quad * \quad \boxed{K = \frac{1}{2}mv^2}$$

doctorphys notes (c. 1980)

The Work-Energy Theorem, or Work-Energy Principle, can be stated as follows. The net force is also called the *resultant force* or simply the *resultant*.

$$\boxed{W(\text{of the resultant force}) = K - K_0 = \Delta K}$$

We will apply the work energy theorem for two cases. An excellent practice in physics, engineering, and math is apply a new technique to a problem for which we already know the answer. This approach gives us confidence in the new technique. Our first application below takes this path.

Application 1. Gravity. Drop a ball from rest at height h above the ground. What is the velocity just before hitting the ground. We can use the kinematic equation

$$2ad = v^2 - v_0^2 ,$$

choosing down as positive with $d = h$, $v_0 = 0$, and $a = g$.

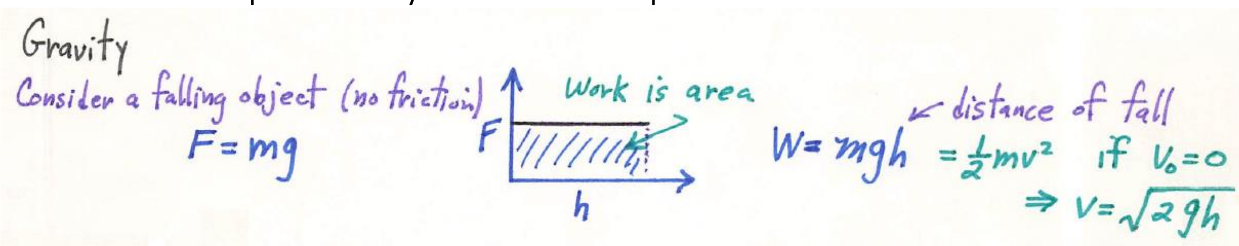
$$2gh = v^2$$

$$v = \sqrt{2gh}$$

Now for the magic of the Work-Energy Theorem. The force is $F = mg$ and distance is $d = h$.

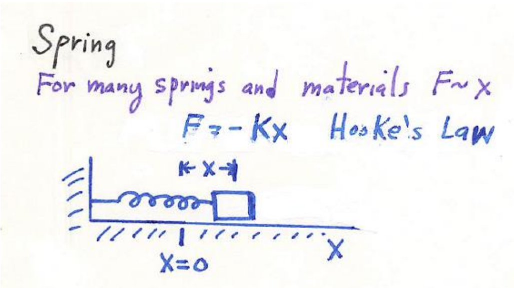
$$W = Fd = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 \quad => \quad mgh = \frac{1}{2}mv^2 \quad => \quad v = \sqrt{2gh}$$

We get the same result! Isn't it easier to remember the work-energy theorem compared to the kinematic formula?
The work can be represented by the area under a plot of force versus distance.



This visualization will be very important in determining work for our next example, the spring.

Application 2. Spring. For the second application we consider an ideal spring where the force necessary to stretch or compress it is proportional to the displacement from equilibrium. This ideal scenario is known as Hooke's Law, named after scientist Robert Hooke (1635-1703).



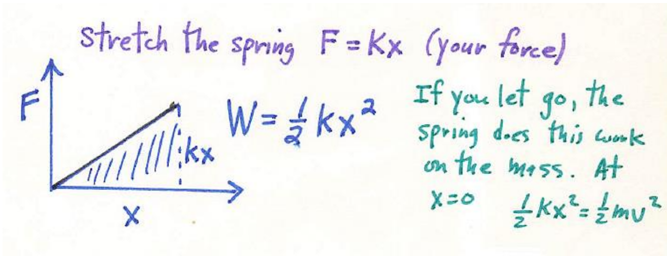
Hooke's Law can be stated as

$$F = -kx,$$

where the force is that by the spring on the mass in the figure. The minus sign is important. If you stretch the spring as shown in the figure along with positive x-direction, the spring force is to the left pulling you back. The distance x is measured from the non-stretch position x = 0, the equilibrium position. If you compress the spring, your x will be negative and the minus sign in Hooke's Law will give you an overall positive spring force, i.e., a force to the right, since the two minus signs cancel.

We will now use the Work-Energy Theorem for the case where we stretch the spring by pulling the mass in the figure to the right and letting go. **There is no friction.** The spring will pull the mass back to the left. What is the speed when the mass reaches the zero point?

We can figure out the work from the area under the force versus distance graph. The graph shows the work you do to pull the mass to the right, stretching the spring. When you release the mass, the spring will do this same work as the mass is pulled back.



The area of a triangle is one-half the base times height:

$$W = \frac{1}{2} x \cdot kx = \frac{1}{2} kx^2 .$$

Releasing the mass from rest,

$$W = \frac{1}{2} kx^2 = \frac{1}{2} mv^2 - \frac{1}{2} mv_0^2 = \frac{1}{2} mv^2$$

since $v_0 = 0$. Therefore,

$$\frac{1}{2} kx^2 = \frac{1}{2} mv^2$$

and the velocity is readily found.

$$kx^2 = mv^2$$

$$v^2 = \frac{k}{m} x^2$$

$$v = \sqrt{\frac{k}{m} x^2}$$

$$v = \sqrt{\frac{k}{m}}\, x$$

In stretching the mass, the applied force changes. Refer to the graph. The area gives the work. With calculus, we would write for the work we do pulling the mass to the right as

$$W = \int_0^x F(x)dx = \int_0^x kxdx = k\int_0^x xdx = k \left. \frac{x^2}{2} \right|_0^x = \frac{1}{2} kx^2 .$$

If integrating over x up to a final value that we also call x bothers you as it used to bother me when I was a student, we can pick a specific maximum x to be equal to A.

$$W=\int_0^A F(x)dx=\int_0^A kxdx=k\int_0^A xdx=k\left.\frac{x^2}{2}\right|_0^A=\frac{1}{2}kA^2$$

Now I would like to demonstrate that the spring pulling on the mass as the mass goes from A to 0, is also the same work. The force that the spring exerts is given by Hooke's Law $F=-kx$.

$$W=\int_0^A F(x)dx=\int_A^0(-kx)dx=-k\int_A^0 xdx=-k\left.\frac{x^2}{2}\right|_A^0=-\frac{1}{2}k(0^2-A^2)=\frac{1}{2}kA^2$$

Hopefully you got this far in your Calculus II course to understand these integrations. The integral gives you the area under the graph, as we mentioned way back in Chapter B.

G7. Power. We are ready to define another physical quantity. Remember, all of these new physical quantities are constructed from length (L), time (T), and mass (M). The power is the rate at which you do work. The figure below provides the definition.

Power

$P=\frac{W}{t}$ constant W (if not, this is the average)

$P=\frac{\Delta W}{\Delta t}$ $\quad P=\lim_{\Delta t\to 0}\frac{\Delta W}{\Delta t}=\frac{dW}{dt}$ (instantaneous power)

Note: $P=\frac{W}{t}=\frac{Fd}{t}=Fv$ (constant W, F, and v)

Units i) $\frac{J}{s}$ = Watt ≡ W

ii) —

iii) hp = 550 $\frac{\text{ft}\cdot\text{lb}}{\text{s}}$

horsepower

The best definition is to go with

$$P=\frac{dW}{dt}.$$

A dimensional analysis is given next.

$$[P]=\left[\frac{dW}{dt}\right]=\frac{[W]}{[t]}=\frac{[F][x]}{[t]}=\frac{[m][a][x]}{[t]}=\frac{M\cdot\frac{L}{T^2}\cdot L}{T}=\frac{ML^2}{T^3}=ML^2T^{-3}$$

Amazing, how different combinations of L, T, and M give a variety of new physical quantities.

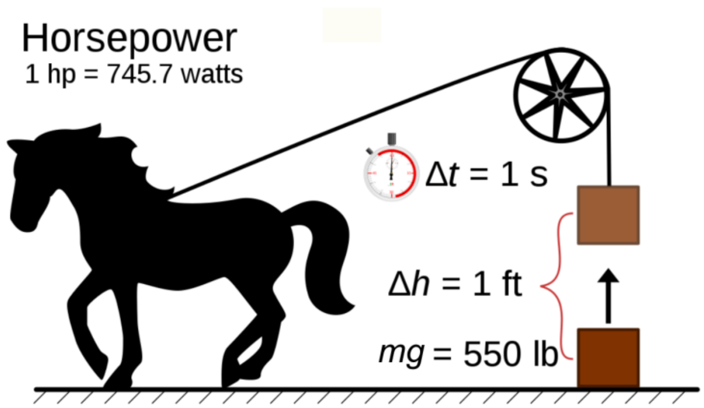
In the MKS Metric System, the unit is named after the Scottish scientist James Watt (1736-1819). Work in the MKS systems is joules. So the watt (W) is a joule/s (J/s). Don't confuse the "W" as a unit with "W" as meaning work. For the unit, you will have a number with it such as 5 watts, i.e., 5 W.

Below is a table with the units in the three main systems use in our text.

	length	time	mass	force	work	power
MKS	m	s	kg	N	N·m = J	J/s = W
cgs	cm	s	g	dyn	dyn·cm = erg	erg/s
British	ft	s	slug	lb	ft·lb	ft·lb/s

In the British system, 1 horsepower (hp) equals 550 ft·lb/s. One way to think of the horsepower is the power to lift 550 lb (2450 N) a distance of 1 foot (30.5 cm) in 1 second. What is 1 horsepower in watts?

$$1\ \text{hp} = 550.0\ \frac{\text{lb}\cdot\text{ft}}{\text{s}} = 550.0\ \frac{\text{lb}\cdot\text{ft}}{\text{s}}\left(\frac{4.44822\ \text{N}}{1\ \text{lb}}\right)\left(\frac{1\ \text{m}}{3.28084\ \text{ft}}\right) = 745.70\ \frac{\text{N}\cdot\text{m}}{\text{s}} = 745.7\ \text{W}$$



Courtesy Sgbeer, Wikimedia
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Change from Original Figure: Added the "g" so that we have mg (weight).

The 550 lb weight has a mass of

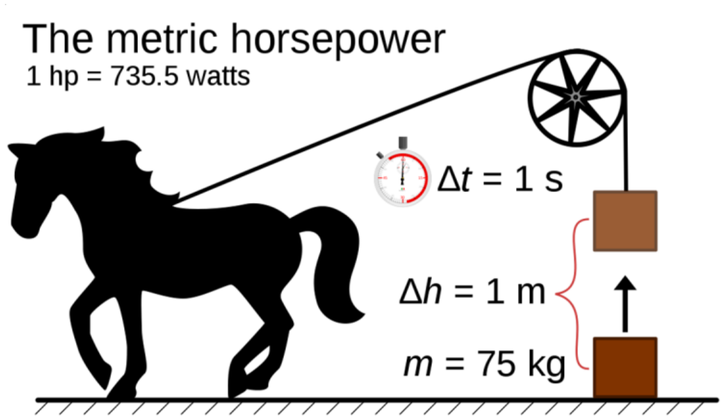
$$550.0\ \text{lb} \rightarrow$$

$$550.0 \cdot \frac{1}{2.20462}\ \text{kg} = 249.48\ \text{kg}$$

The distance of 1 ft is equal to 0.3048 meter. The work done here in 1 second is then

$$W = (249.48\ \text{kg}\cdot\text{wt})(0.3048\ \text{m}) = 76.04\ \text{kg}\cdot\text{wt}\cdot\text{m}$$

If you nudge this value to 75 you get the "metric horsepower."



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NOTE: The metric horsepower is slightly less than the regular horsepower.

$$1\ \text{hp} = 745.70\ \text{W}$$

$$1\ \text{hp}_{\text{metric}} = 745.70\frac{75.00}{76.04} = 735.5\ \text{W}$$

Colin and Mom, Courtesy Patrick W., flickr. [License: Attribution 2.0 Generic.]
Photo taken March 5, 2006. Slight contrast enhancement on photo.

Earlier we found that mom pulled with 18 N at 40°. What is the power if the sled moves at a constant speed of 1 meter per second.

$$P = \frac{W}{t} = \frac{(F\cos\theta)x}{t} = Fv\cos\theta$$

$$P = Fv\cos\theta = 18\ \text{N}\cdot 1\frac{\text{m}}{\text{s}}\cdot\cos 40^\circ = 13.79\ \text{W} = 14\ \text{W}$$

$$P = 13.79\ \text{W}\,\frac{1\ \text{hp}}{745.7\ \text{W}} = 0.02\ \text{hp}$$

We have made a neat discovery in passing – the appearance of the velocity in the power formula! Take a constant force along the direction of motion and constant speed. Then,

$$P = \frac{W}{t} = \frac{Fx}{t}$$

$$\boxed{P = Fv}$$

You might have noticed that this formula appeared in one of our figures above.