Physics I with Calculus, Prof. Ruiz (Doc), UNC-Asheville (1978-2021), DoctorPhys on YouTube
Chapter B. Linear Kinematics. Prerequisite: Calculus I. Corequisite: Calculus II.

B1. Kinematics, Speed, and Units: Units Tell You What to Do. The descriptive study of motion without analyzing the forces that cause the motion is called *kinematics*. Therefore, units associated with kinematics include distance, time, and speed. The relationship among these units are not derived by any fundamental laws of physics. The laws of physics that tell us how things move will involve forces. We call that topic of physics *dynamics*. Let's illustrate how units in kinematics follow from common-sense definitions.

We return to distance, time, and speed. What is your distance traveled in meters if you travel at a speed of five meters per second (5 m/s) for 2 seconds (2 s). The answer is

$$5\frac{\text{m}}{\text{s}}\cdot 2\text{ s} = 10\text{ m}.$$

Over the years of my teaching students would tell me they learned the formula distance equals rate times time. I did not like hearing about the word rate. The formula they are talking about is one I like to write as

$\text{distance} = \text{velocity}\cdot\text{time}$, which with symbols, is

$$d = vt.$$

Such a formula is a kinematic formula, following from the units themselves. A typical student would solve the above problem by writing down the formula and then plugging in the numbers as follows:

$d = vt$ => $d = 5\cdot 2 = 10\text{ m}$ since $v = 5$ and $t = 2$.

I would prefer for now, that you forget the formula and use cancellation of units to get your answer. Write down one of the given values, say the speed of $5\frac{\text{m}}{\text{s}}$. You want meters. Therefore, you have to cancel that second (s) in the denominator. You use your other "given" data to do the job.

$$5\frac{\text{m}}{\text{s}}\cdot 2\text{ s} = 10\text{ m}.$$

You did not even need to know the formula! Suppose you wrote down 2 s to start with? You use the other "given" data to transform the seconds into meters.

$$2\text{ s}\cdot 5\frac{\text{m}}{\text{s}} = 10\text{ m}$$

Again, no formula. The units tell you what to do. And using the units, you are less prone to error. I have seen many students grab the formula and plug in numbers without thinking too much about units and going astray. That said, we will arrive at kinematic formulas in this chapter. You can consider these as being derived and we will use various algebraic methods.

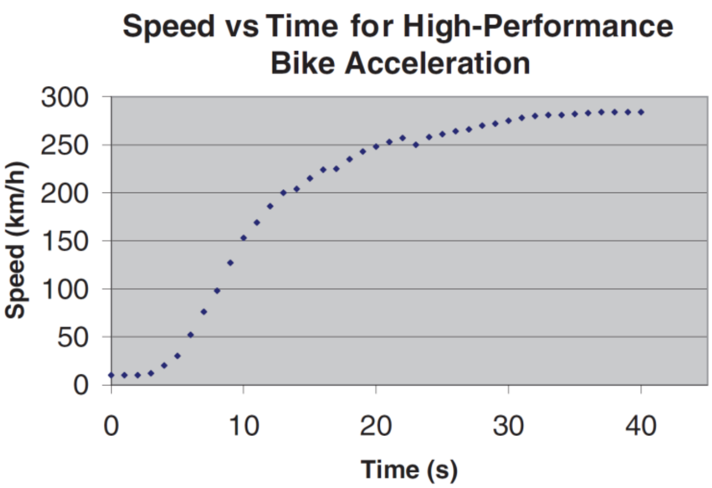
B2. Acceleration. Another physical quantity that arises naturally in kinematics is acceleration. Back in 2009, I was the first author to publish an article in *The Physics Teacher* on using YouTube to teach physics. The paper dealt with kinematics:

Michael J. Ruiz, "Kinematic Measurements from YouTube Videos,"
The Physics Teacher **47**, 200-203 (April 2009).

One of the examples in the paper is data from a BMW K1200S Motorcycle, a high-speed bike produced from 2003 to 2008 in Europe and the United States. Such a bike appears below.

BMW K1200S
Sungwon Kim, Flickr
[Creative Commons (CC BY-NC-ND 2.0)]

The data we will use came from a YouTube video posted in 2006 and published in the above paper. Some have questioned the authenticity of the actual video, but we can still illustrate the physics concepts with the data as given.



A plot of the speed in kilometers per hour (km/h) against time in seconds (s) is given at the left. The figure comes from a YouTube video.

The data was taken from the speedometer seen in the video. The video player gives the time in seconds. Note the drop in speed in two places as gears were shifted.

The bike accelerates from rest to a speed of 280 kilometers/hour in 40 seconds. The acceleration is another physics quantity derived from two of our three base units: distance and time. We do not need mass for this one. To illustrate the concept, consider an object that starts from rest and then every second it picks up 2 m/s in speed. Therefore, after 1 second, it is going 2 m/s; after 2 seconds it is going 4 km/h, and so on. The table below lists the change in speed over a few seconds.

Time (s)	Speed (m/s)
0	0
1	2
2	4
3	6
4	8
5	10

The acceleration is defined as the change in speed per time interval. In our case, the increasing change is constant. Let v represent the speed. The letter v stands for the magnitude of the velocity, which is the speed. We consider a moving object along a straight line, i.e., no change in direction.

Let the change in speed be expressed as delta v: Δv, where the delta is the Greek letter Δ. The Greek lowercase for delta is the symbol δ. Greek letters are often used in physics and mathematics. The corresponding change of time is Δt. To get the change in speed per time interval, we divide as per always means division. The acceleration is

$$a = \frac{\Delta v}{\Delta t},$$

which in general gives an average acceleration because the acceleration can change from moment to moment. In our case, it does not. We are looking at a case of constant acceleration over our time interval. Plugging in the numbers for the interval of the entire 5 seconds,

$$a = \frac{\Delta v}{\Delta t} = \frac{10\text{ m/s} - 0}{5\text{ s} - 0} = \frac{10\text{ m/s}}{5\text{ s}} = 2\frac{\text{m}}{\text{s}} \cdot \frac{1}{\text{s}} = 2\frac{\text{m}}{\text{s}^2},$$

where we keep the units. Note that in the denominator we have seconds multiplying seconds. We write that as seconds squared. Think of a change, i.e., the delta, as writing down the final value and subtracting the initial value. A change in velocity is then written as $\Delta v = v_{\text{final}} - v_{\text{initial}}$, which we can abbreviate as $\Delta v = v_{\text{f}} - v_{\text{i}}$. The corresponding change in time is then $\Delta t = t_{\text{f}} - t_{\text{i}}$. The general equation for the acceleration follows as

$$a = \frac{\Delta v}{\Delta t} = \frac{v_{\mathrm{f}} - v_{\mathrm{i}}}{t_{\mathrm{f}} - t_{\mathrm{i}}} .$$

But in real life we want the Δt to be very small so that we obtain an accurate value of the acceleration at a given instant. Technically, we want Δt to approach zero. You have encountered this idea in calculus. Remember that this physics course has calculus as a prerequisite. In calculus we write

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} ,$$

where lim stands for limit. Note that as $\Delta t \to 0$, we also have $\Delta v \to 0$, giving a finite ratio. This point is important because we never want to divide by zero. We are taking the limit here as Δt approaches 0. Remember our definition of velocity as distance divided by time? By analogy, we can write

$$v = \frac{\Delta x}{\Delta t} = \frac{x_{\mathrm{f}} - x_{\mathrm{i}}}{t_{\mathrm{f}} - t_{\mathrm{i}}} \quad \text{and} \quad v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} ,$$

where the distance variable is x. An excellent visualization of these quantities is found in the concept of slope. When we plotted height (distance) against time in our last chapter, the slope gave the growth rate, a speed. We investigate the slope more in the next section.

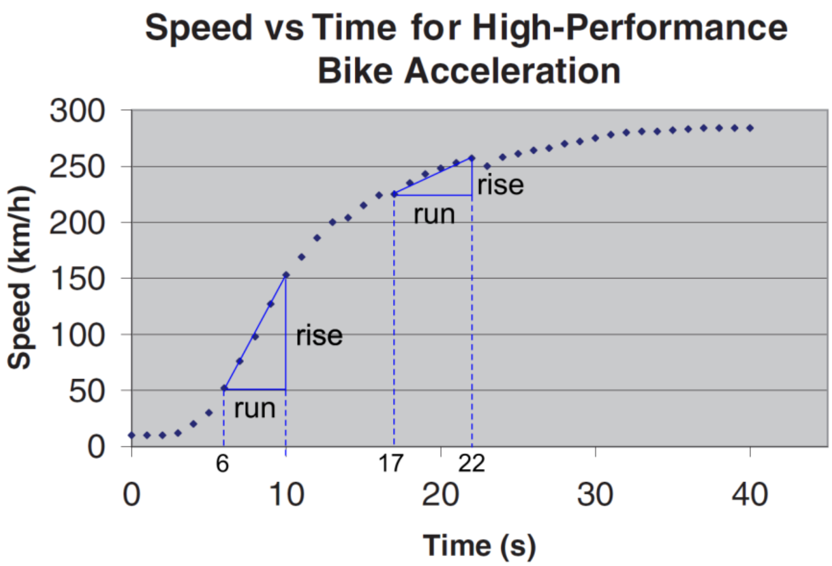
Before we leave this section we can derive an important formula we will need later. We consider a case of constant acceleration $a = \frac{\Delta v}{\Delta t} = \frac{v_{\mathrm{f}} - v_{\mathrm{i}}}{t_{\mathrm{f}} - t_{\mathrm{i}}}$. Now let the final velocity be $v_{\mathrm{f}} = v$ and the initial velocity be $v_i = v_0$. Then let the final time be $t_{\mathrm{f}} = t$ and the initial time $t_i = 0$. In other words, we start the timer at the initial time. Then

$$a = \frac{\Delta v}{\Delta t} = \frac{v_{\mathrm{f}} - v_{\mathrm{i}}}{t_{\mathrm{f}} - t_{\mathrm{i}}} = \frac{v - v_o}{t - 0} ,$$

which leads to $at = v - v_o$ and the very important formula

$$v = v_0 + at .$$

B3. The Slope of a Velocity vs Time Graph. The key to understanding how calculus relates to kinematics is the velocity versus time graph. Note that at times the word velocity refers to the magnitude of the velocity vector, i.e., velocity can be used for speed in practice.



Applying this concept to our graph the speed versus time, the slope gives the acceleration. The units corresponding to the graph at the left are

$$\frac{\text{km/h}}{\text{s}} = \frac{\text{km}}{\text{h}\cdot\text{s}} .$$

If you prefer, you can also write

$$\frac{\text{km}}{\text{h}\cdot\text{s}} = \frac{\text{km}}{3600\ \text{s}\cdot\text{s}} = \frac{1}{3600}\frac{\text{km}}{\text{s}^2}$$

since there are 3600 seconds (3600 s) in one hour (h). I like to leave things as $\frac{\text{km}}{\text{h}\cdot\text{s}}$. The acceleration is greatest where the slope is steepest. Look at the time between about 6 seconds and 10 seconds, where the speed goes from 50 km/h to 150 km/h. The acceleration over this range is

$$a = \frac{\text{rise}}{\text{run}} = \frac{\Delta v}{\Delta t} = \frac{v_\text{f} - v_\text{i}}{t_\text{f} - t_\text{i}} = \frac{150\ \text{km/h} - 50\ \text{km/h}}{10\ \text{s} - 6\ \text{s}} = \frac{100\ \text{km/h}}{4\ \text{s}} = 25\frac{\text{km}}{\text{h}\cdot\text{s}} .$$

Every second we pick up an additional 25 km/h in the speed. For the slope between 17 s and 22 s, the values reading the graph as best as we can, is

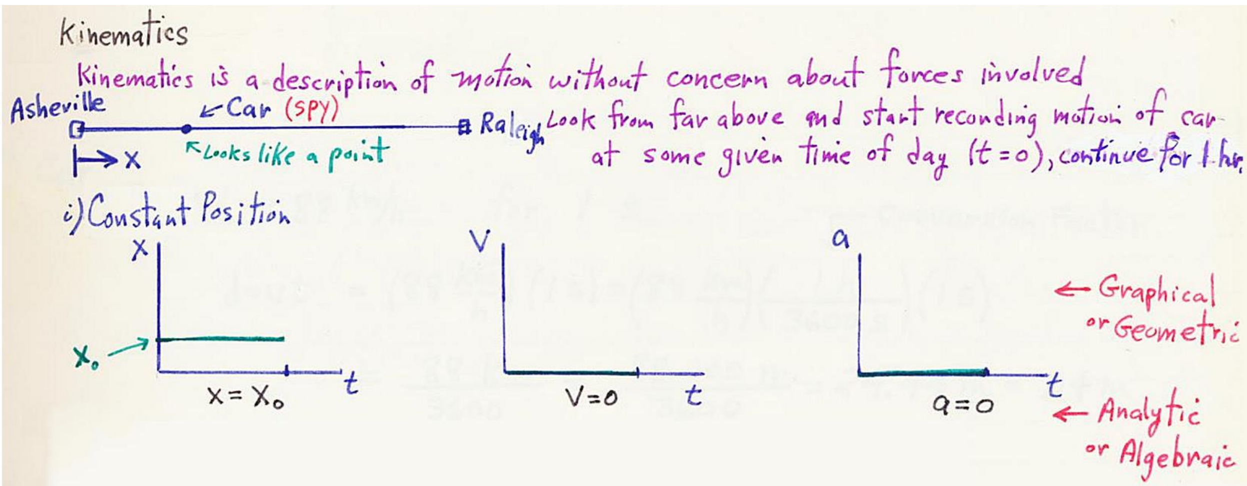
$$a = \frac{\text{rise}}{\text{run}} = \frac{\Delta v}{\Delta t} = \frac{v_\text{f} - v_\text{i}}{t_\text{f} - t_\text{i}} = \frac{260\ \text{km/h} - 225\ \text{km/h}}{22\ \text{s} - 17\ \text{s}} = \frac{35\ \text{km/h}}{5\ \text{s}} = 7\frac{\text{km}}{\text{h}\cdot\text{s}} .$$

In this time interval we pick up a reduced 7 km/h every second. When we reach over 35 s, the graph flattens out. A horizontal flat line means no slope. There is no rise. The acceleration is then zero and the velocity constant. Note that units for a can always be expresses as $\frac{\text{m}}{\text{s}\cdot\text{s}} \equiv \frac{\text{m}}{\text{s}^2}$.

If we were to make a graph of the acceleration as a function of time, we would have what they call the *derivative* in calculus class. The new graph is a *derived graph* where we plot the slope of the original graph at all points.

B4. Four Equations of Linear Kinematics. Here are notes I made in 1978 when I first started teaching at UNC Asheville. We will consider a car traveling east, but first watch a car at rest. Imagine looking at the car from a position above the road, like from the sky looking down.

Case 1. Car at Rest.

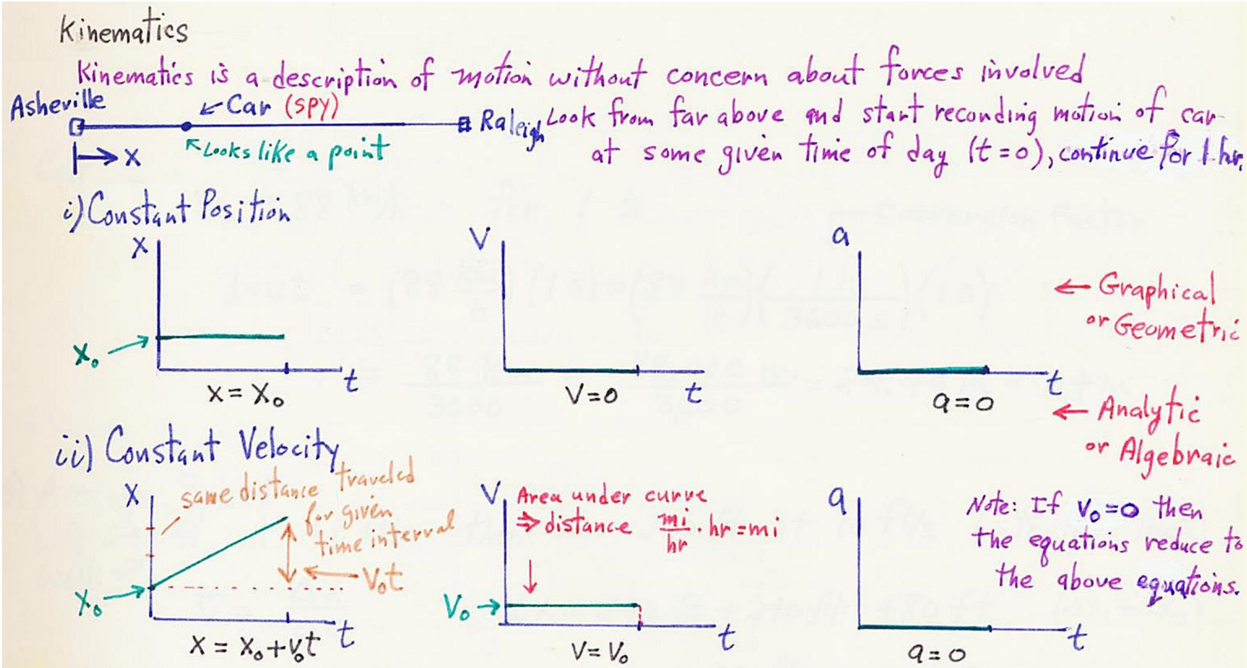


The car does not move and is located a distance x_0 from Asheville. Therefore, plotting distance from Asheville against time gives a horizontal line at $x = x_0$. The velocity is zero since the car is at rest parked. Therefore, the velocity plot against time is also a horizontal line, where the speed is zero: $v = 0$. Similarly, the car is not accelerating. So the plot or acceleration against time is also a horizontal line at zero: $a = 0$. Note the three formulas written under the graphs:

$$x = x_0,\ v = 0, \text{ and } a = 0.$$

Technically, the horizontal lines in the graphs go forever. These formulas are our kinematic formulas for this case where the car is at rest. We will now proceed to move to more general cases of motion. First, we will consider a car traveling at constant velocity: straight line motion with constant speed.

Case 2. Car with Constant Velocity.



This case has a car at position $x = x_0$ initially, but is now traveling at that moment and beyond at a constant velocity $v = v_0$. There is no acceleration since there is no change in velocity. Therefore $a = 0$ at all times. The secret here is to start with the velocity graph. Focusing on the units, the area of the rectangle shown represents the distance traveled. We can see this feature easily by picking numbers. Say $v_0 = 2\dfrac{\text{m}}{\text{s}}$ and a time for the horizontal green line as $t = 5\text{ s}$. Don't worry about our use of t to represent the axis and also this specific time for the horizontal green extent. If you want, you can change the t representing the axis to $t\text{-axis}$. The variable t represents any time along the axis. It is the general time variable.

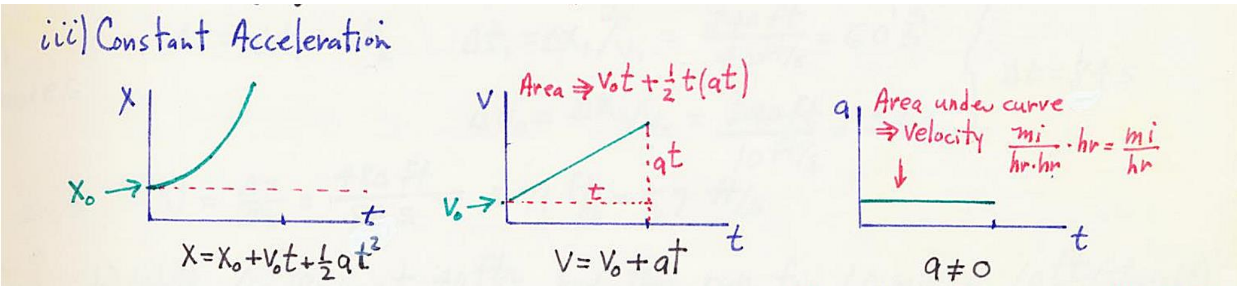
The distance traveled for the car going $v_0 = 2\dfrac{\text{m}}{\text{s}}$ for $t = 5\text{ s}$ is given by playing with the units. You must multiply: $2\dfrac{\text{m}}{\text{s}} \cdot 5\text{ s} = 10\text{ m}$. This value is represented by the area under the green line. We obtain the area of this rectangle by multiplying length times width. The length represents the $t = 5\text{ s}$ and the width of the string is represented by $v_0 = 2\dfrac{\text{m}}{\text{s}}$. Area represents distance? That is weird, but correct. Therefore, the distance traveled is given by $10\text{ m} = 2\dfrac{\text{m}}{\text{s}} \cdot 5\text{ s} = v_0 t$. But since we start out at $x = x_0$, the formula for the distance from Asheville as the reference is

$$x = x_0 + v_0 t\ .$$

We can graph this equation for $x = x(t)$. It is a straight line with slope v_0 and vertical intercept x_0. Remember from algebra the basic linear equation $y = mx + b$. Now, the three formulas listed under the graphs are respectively $x = x_0 + v_0 t$, $v = v_0$, and $a = 0$. Note that these formulas for the case where $v = v_0 = 0$ reduce to our previous formulas for the car at rest: $x = x_0$, $v = 0$, and $a = 0$.

Such reduction of formulas is one of my favorite aspects of physics. The more general formulas reduce to simpler formulas when the situation simplifies. Our next case is even more general: one with constant acceleration that is nonzero.

Case 3. Car with Constant Acceleration (Nonzero).



We start again with the middle graph. The velocity formula is now $v = v_0 + at$. The convention is not to write $a = a_0$, but just a.The formula follows from the units. Remember from our last section that acceptable units for a are $\frac{\text{m}}{\text{s}\cdot\text{s}} \equiv \frac{\text{m}}{\text{s}^2}$. As an example, take $a = 2\ \frac{\text{m}}{\text{s}^2}$ and $t = 5\ \text{s}$. The gain in velocity after the five seconds is

$$\Delta v = 2\ \frac{\text{m}}{\text{s}^2} \cdot 5\ \text{s} = at = 10 \frac{\text{m}}{\text{s}}.$$

So that final speed is whatever we started with plus this additional gain:

$$v = v_0 + at .$$

Finally, to give the distance traveled, we use the area trick with the middle graph. We need the area of a trapezoid, which we break down into a triangle that sits on a rectangle:

$$\text{area of rectangle} = v_0 t \text{ (length times width)},$$

$$\text{area of triangle} = \frac{1}{2} t \cdot at = \frac{1}{2} at^2 \text{ (one half the base times the height)}.$$

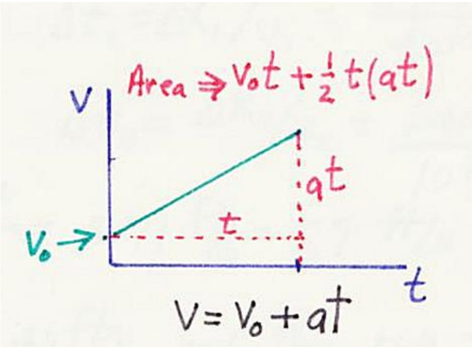
We need to add these together and then add the sum to the initial starting position x_0. The result is

$$x = x_0 + v_0 t + \frac{1}{2}at^2 .$$

We have two main formulas, which we list in the table below. Our variables are x, v, a, and t with initial conditions x_0 and v_0. We list the variable that is missing in each equation. This feature is good because we usually are missing some data when given a problem to solve.

$x = x_0 + v_0 t + \frac{1}{2}at^2$	There is no v in the equation.
$v = v_0 + at$	There is no x in the equation.
???	How about an equation with no a?
???	How about an equation with no t?

It is nice to have four equations, where one of our variables is missing in each. Let's go for one without the acceleration a.



We can use the formula for a trapezoid instead of the rectangle and triangle. The two altitudes are

$$v = v_0 \text{ and } v = v_0 + at .$$

We take the average and multiply by the base t.

$$\text{area} = \frac{1}{2}(v_0 + v)t .$$

The alternate distance formula is then

$$x = x_0 + \frac{1}{2}(v_0 + v)t .$$

This result brings us to the following table.

$x = x_0 + v_0 t + \frac{1}{2}at^2$	no v
$v = v_0 + at$	no x
$x = x_0 + \frac{1}{2}(v_0 + v)t$	no a
???	How about an equation with no t?

We get the last equation by using the second and third equations to eliminate t. But before we do this manipulation, I would like to introduce the parameter $d = x - x_0$, the distance traveled from the starting point x_0. Then, there is less to write down. Take the second equation $v = v_0 + at$ in the form $t = \frac{v - v_0}{a}$ and substitute into the third equation in the form $d = \frac{1}{2}(v_0 + v)t$:

$$d = \frac{1}{2}(v_0 + v)t = \frac{1}{2}(v_0 + v)\frac{(v - v_0)}{a}.$$

Remember from algebra that $(a+b)(a-b) = a^2 - b^2$. Therefore,

$$d = \frac{1}{2}(v_0 + v)\frac{(v - v_0)}{a} = \frac{1}{2}(v + v_0)\frac{(v - v_0)}{a} = \frac{v^2 - v_0^2}{2a}.$$

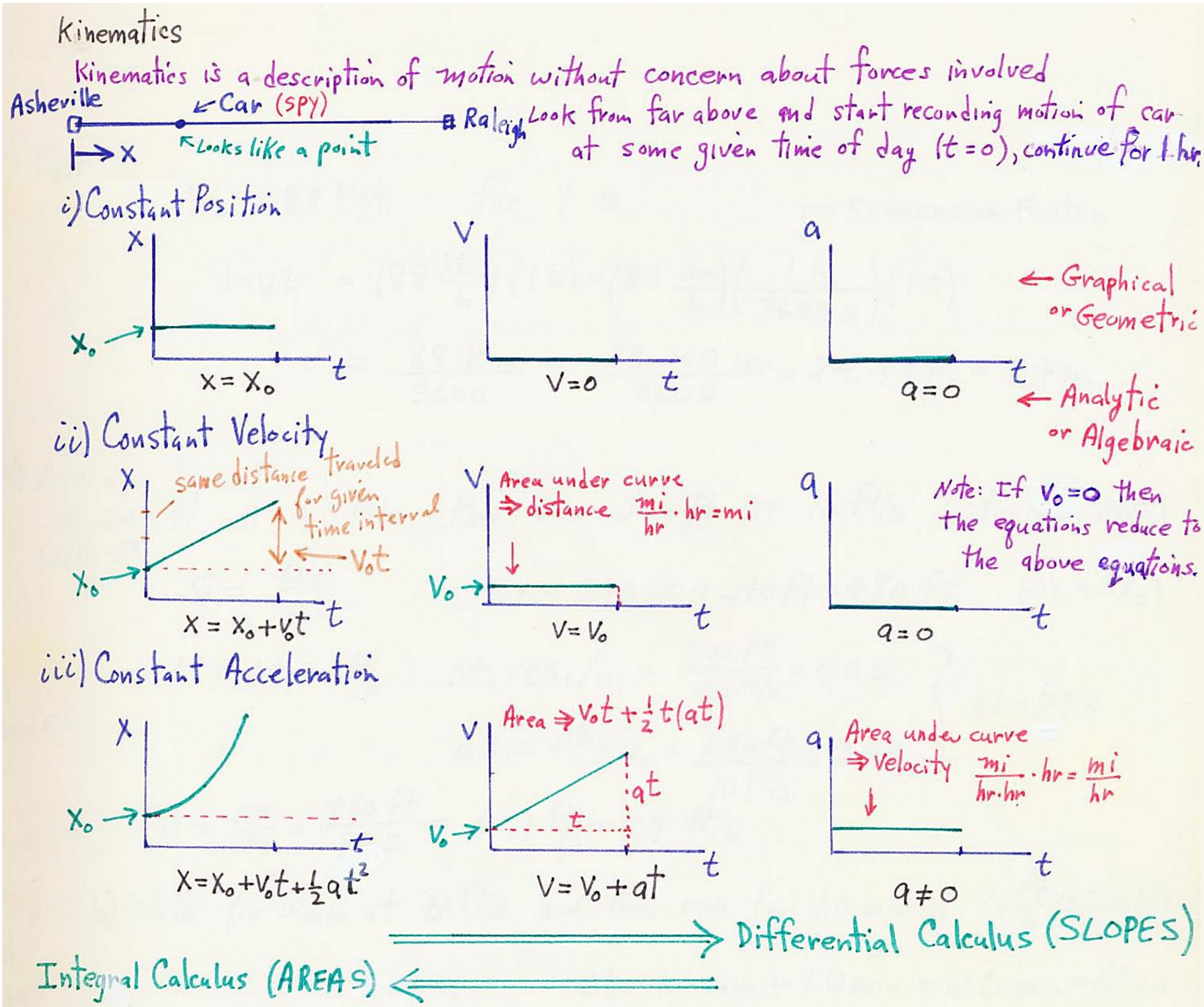
We now have the four desired equations.

$x = x_0 + v_0 t + \frac{1}{2}at^2$	no v
$v = v_0 + at$	no x
$x = x_0 + \frac{1}{2}(v_0 + v)t$	no a
$x = x_0 + \frac{v^2 - v_0^2}{2a}$	no t

I like to remember the last one $d = \frac{v^2 - v_0^2}{2a}$ as $2ad = v^2 - v_0^2$. You can pick your own favorite way to remember these four important and basic kinematic equations. Now for the big summary and connection with calculus. See the next chart.

Moving from one graph to the next one on the right => differential calculus (slopes).
Moving from one graph to the next one on the left => integral calculus (areas).

But do not worry if the integral calculus on the following page is strange to you. Integral calculus is taught in Calculus II and that course is a corequisite for this course. So you can skip over the integral calculus you see on the next page for now.



$$x = x_o + v_o t + \frac{1}{2}at^2 \quad => \quad \frac{dx}{dt} = 0 + v_o + \frac{1}{2}a(2t) \quad => \quad v = v_o + at$$

$$x - x_o = v_o t + \frac{1}{2}at^2 \quad <= \quad \int_{x_o}^{x} \frac{dx}{dt}dt = \int_0^t v dt = \int_0^t (v_o + at)dt \quad <= \quad v = v_o + at$$

The same applies to the velocity and acceleration graph pair.

$$v = v_o + at \quad => \quad \frac{dv}{dt} = 0 + a = a \quad => \quad a = a$$

$$v = v_o + at \quad <= \quad \int_{v_o}^{v} \frac{dv}{dt}dt = \int_0^v a dt \quad <= \quad a = a$$

B5. Real-World Kinematics. In preparing this course, I explored a real-world example from a YouTube video and published the exercise.

Michael J. Ruiz, "Speed, acceleration, and distance plots from a racecar Speedometer (0 – 300 km^{-1})" *Physics Education* **56**, 055035 (September 2021).

I would like to proceed along these lines with data from a Camaro doing a test run where it accelerated from 0 to 300 km/h in 40 seconds. See the photo below for three racing cars. The two cars on the left are Camaros.

Three cars racing, Courtesy Photographer Zach Catanzareti. Used by permission.
https://www.flickr.com/photos/58980992@N03/albums/

The data will come from

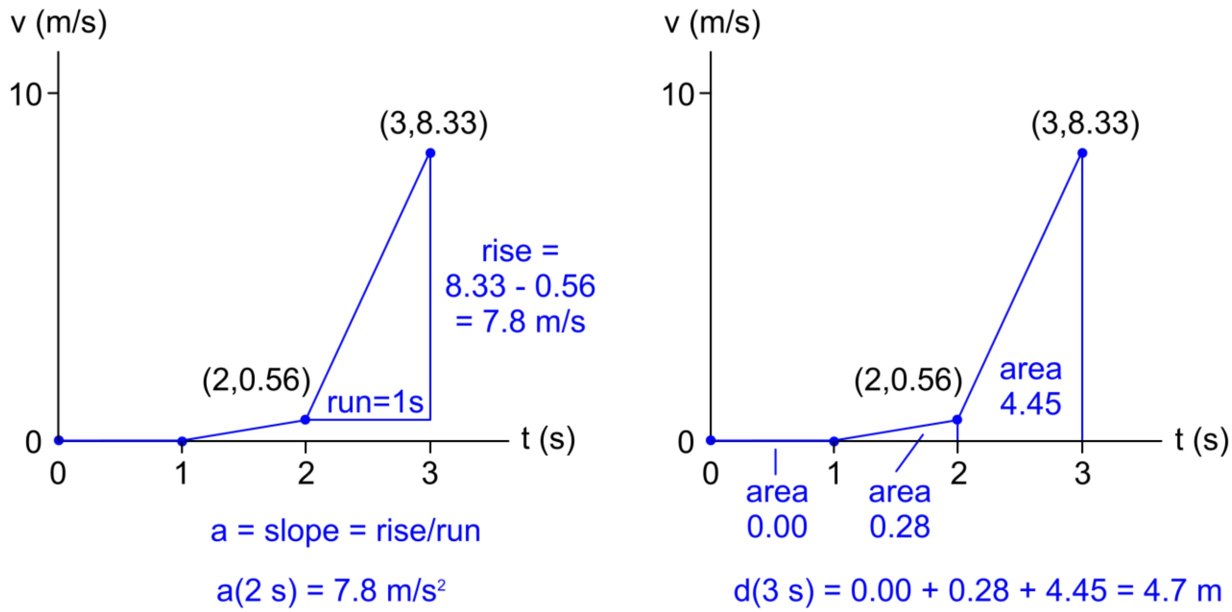
Alharmoodi H 2019 2018 Camaro ZL1 Stock from 0 – 302 km/h YouTube
https://www.youtube.com/watch?v=jf3fEcUiB0I

The results from noting the speedometer every second are shown in the table below.

Table 1. Kinematic data for the accelerating Camaro.

Time **s**	**Velocity** **km h^{-1}**	**Velocity** **m s^{-1}**
0	0	0.00
1	0	0.00
2	2	0.56
3	30	8.33
4	46	12.78
5	76	21.11
6	99	27.50
7	112	31.11
8	138	38.33
9	153	42.50
10	163	45.28
11	176	48.89
12	188	52.22
13	198	55.00
14	208	57.78
15	216	60.00
16	224	62.22
17	231	64.17
18	237	65.83
19	242	67.22
20	247	68.61
21	251	69.72
22	256	71.11
23	260	72.22
24	264	73.33
25	267	74.17
26	271	75.28
27	274	76.11
28	277	76.94
29	281	78.06
30	284	78.89
31	287	79.72
32	289	80.28
33	292	81.11
34	293	81.39
35	295	81.94
36	297	82.50
37	298	82.78
38	299	83.06
39	300	83.33
40	301	83.61
41	302	83.89

The slope for each second gives the acceleration at each second. The area for each section gives the distance traveled. These results are approximate since we are simplifying the graph as straight line segments. Sample slope and area calculations are shown below.



The velocity (speed) against time.

Slope at zero seconds is taken from the slope between t = 0 and t = 1. The slope is zero. The slope at 1 s is calculated from the slope between 1 and 2. The slope is

$$\text{slope at 1 s} = \frac{\text{rise}}{\text{run}} = \frac{0.56-0}{2-1} = 0.56\ \frac{\text{m}}{\text{s}^2}$$

$$\text{slope at 2 s} = \frac{\text{rise}}{\text{run}} = \frac{8.33-0.56}{3-2} = 7.8\ \frac{\text{m}}{\text{s}^2}$$

For the areas, the area in the interval from 0 to 1 s is zero.

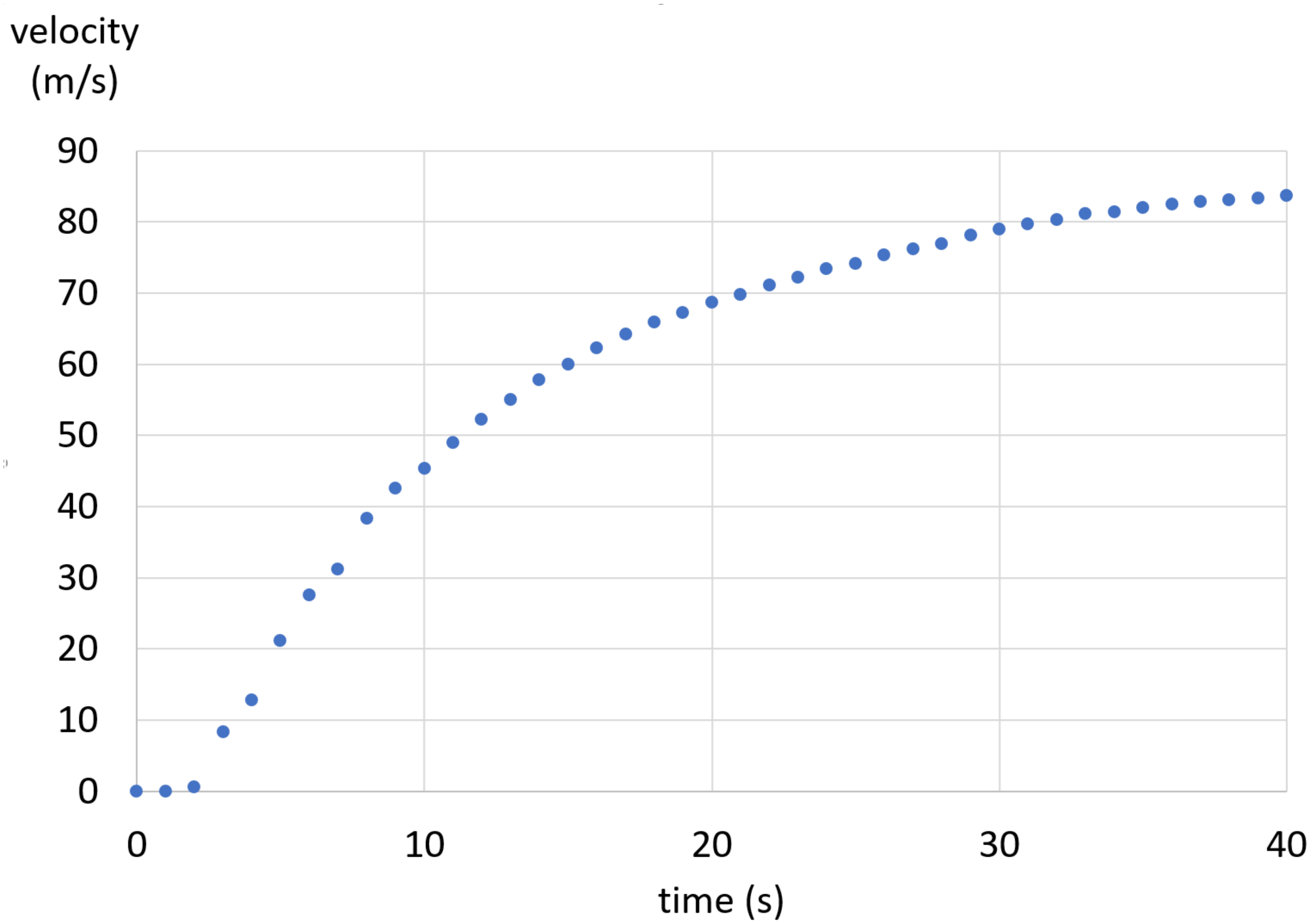
$\text{area between 0 s and 1 s} = 0$ => d(1) = 0, i.e., the distance gained by 1 s.

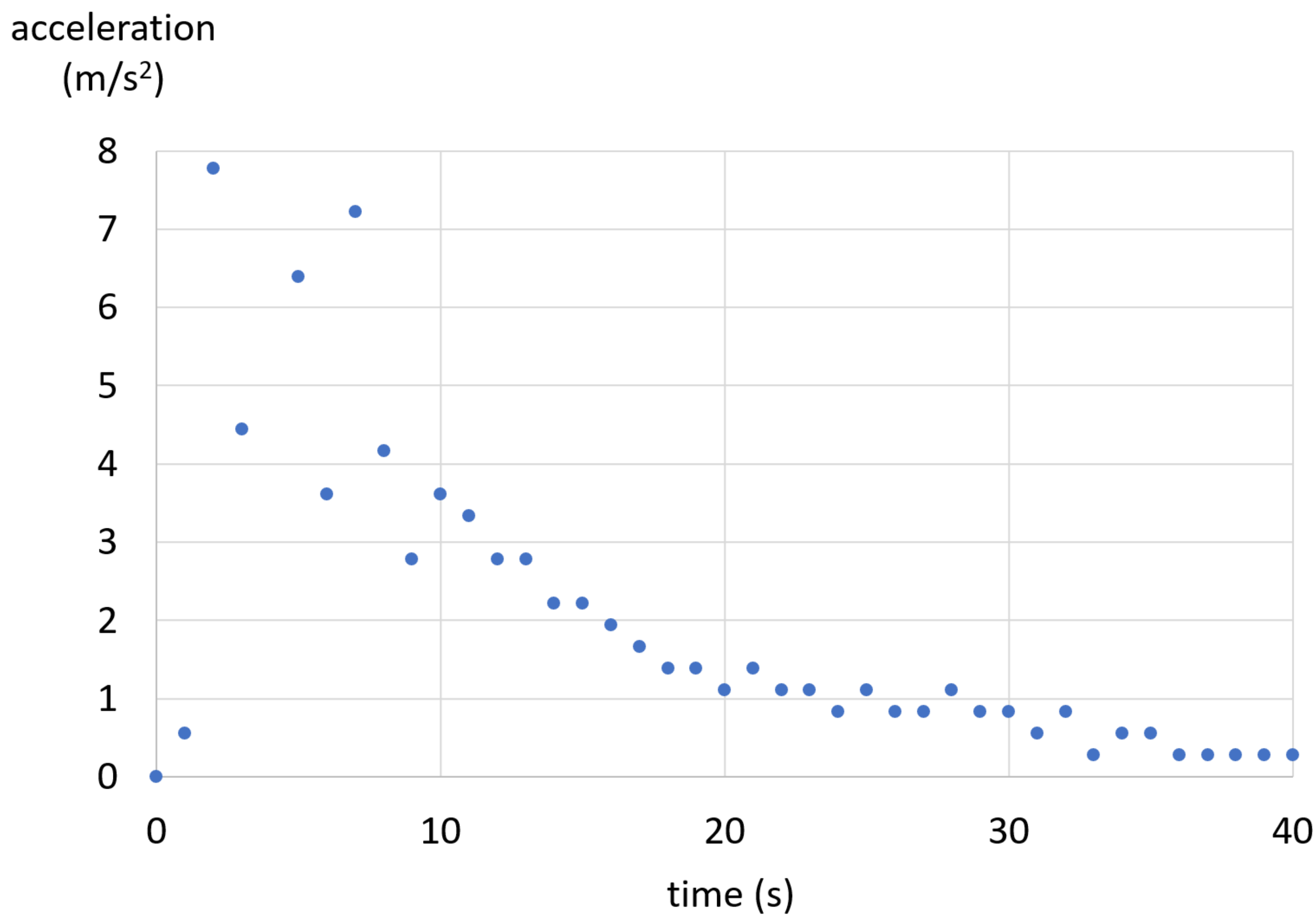
$\text{area between 1 s and 2 s} = \frac{1}{2} 0.56 \cdot 1 = 0.28\ \text{m}$ => d(2) = 0.28 m.

$\text{area between 2 s and 3 s} = \frac{1}{2}(0.56 + 8.33) \cdot 1 = 4.45\ \text{m}$ => d(3) = 4.7 m.

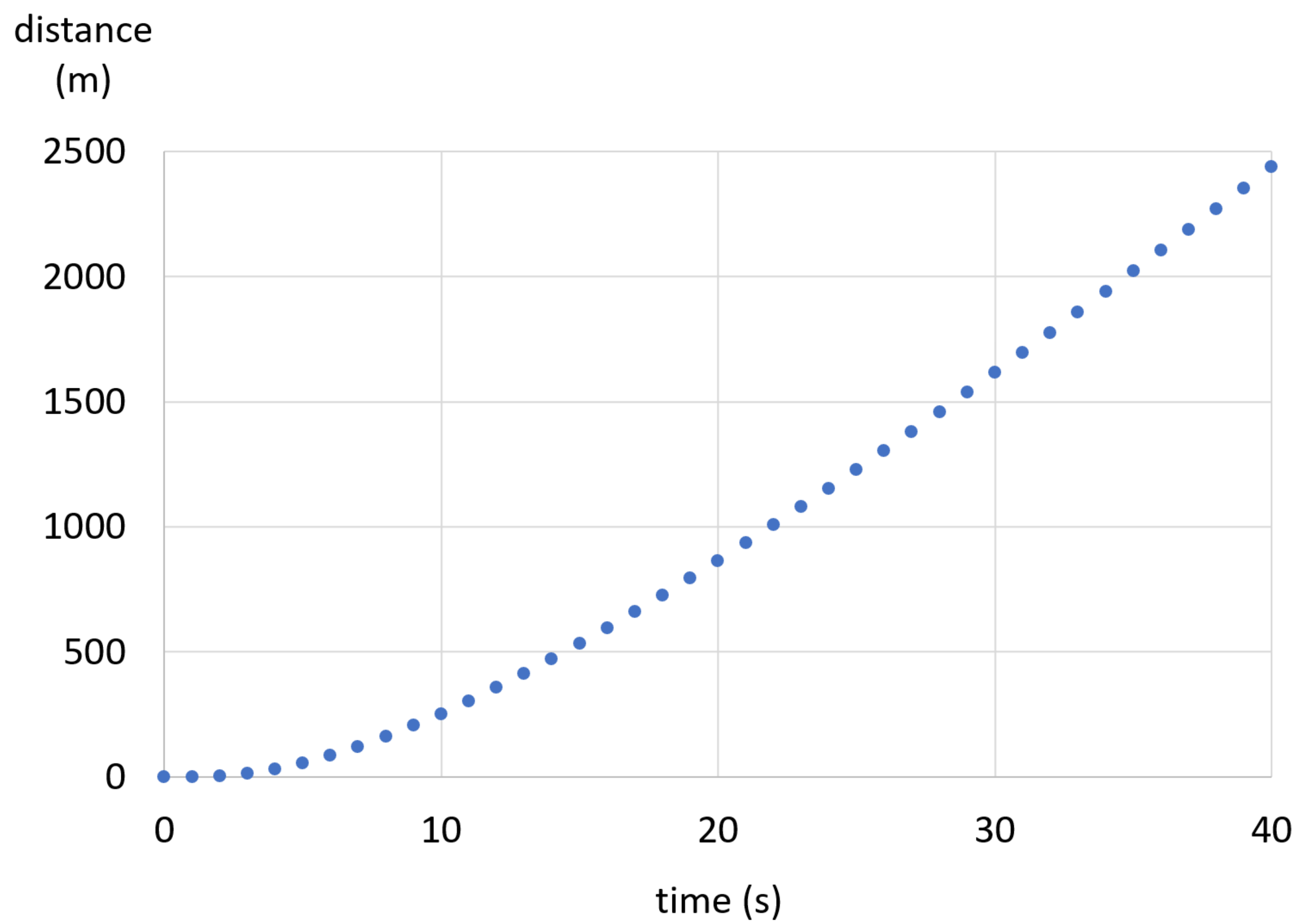
Time s	Velocity km h^{-1}	Velocity m s^{-1}	Acceleration m s^{-2}	Distance m
0	0	0.00	0.00	0.00
1	0	0.00	0.56	0.28
2	2	0.56	7.78	4.72
3	30	8.33	4.44	15.28
4	46	12.78	8.33	32.22
5	76	21.11	6.39	56.53
6	99	27.50	3.61	85.83
7	112	31.11	7.22	120.56
8	138	38.33	4.17	160.97
9	153	42.50	2.78	204.86
10	163	45.28	3.61	251.94
11	176	48.89	3.33	302.50
12	188	52.22	2.78	356.11
13	198	55.00	2.78	412.50
14	208	57.78	2.22	471.39
15	216	60.00	2.22	532.50
16	224	62.22	1.94	595.69
17	231	64.17	1.67	660.69
18	237	65.83	1.39	727.22
19	242	67.22	1.39	795.14
20	247	68.61	1.11	864.31
21	251	69.72	1.39	934.72
22	256	71.11	1.11	1006.39
23	260	72.22	1.11	1079.17
24	264	73.33	0.83	1152.92
25	267	74.17	1.11	1227.64
26	271	75.28	0.83	1303.33
27	274	76.11	0.83	1379.86
28	277	76.94	1.11	1457.36
29	281	78.06	0.83	1535.83
30	284	78.89	0.83	1615.14
31	287	79.72	0.56	1695.14
32	289	80.28	0.83	1775.83
33	292	81.11	0.28	1857.08
34	293	81.39	0.56	1938.75
35	295	81.94	0.56	2020.97
36	297	82.50	0.28	2103.61
37	298	82.78	0.28	2186.53
38	299	83.06	0.28	2269.72
39	300	83.33	0.28	2353.19
40	301	83.61	0.28	2436.94
41	302	83.89	0.00	2520.83

The graphs follow.





Note that the acceleration is a little erratic at the beginning of the trip. Recall your own experience in a car that accelerates from rest. Things can be jerky. Then, things smooth out.



See the next figure for the relationship among distance, speed, and acceleration within the framework of calculus which we have already discussed. Differential calculus involves plotting the slope of a function. Moving to the right in the figure shows the derivatives. Integral calculus gives the area under the graph, illustrated in figure 7 as moving to the left.