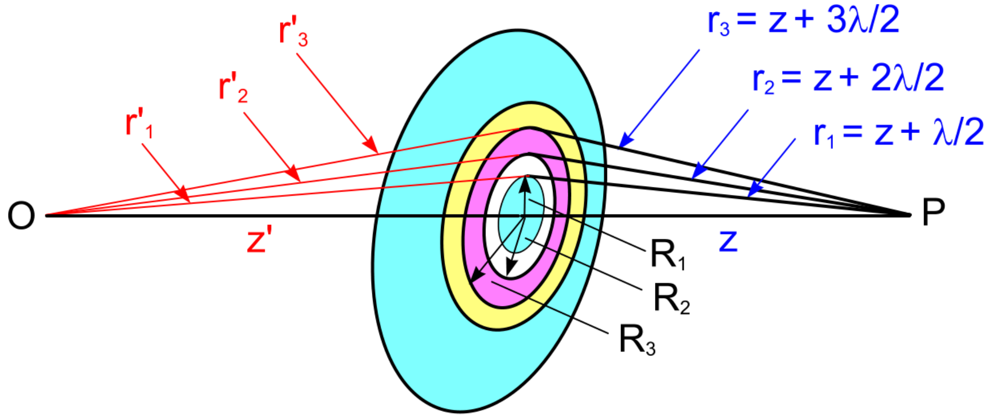
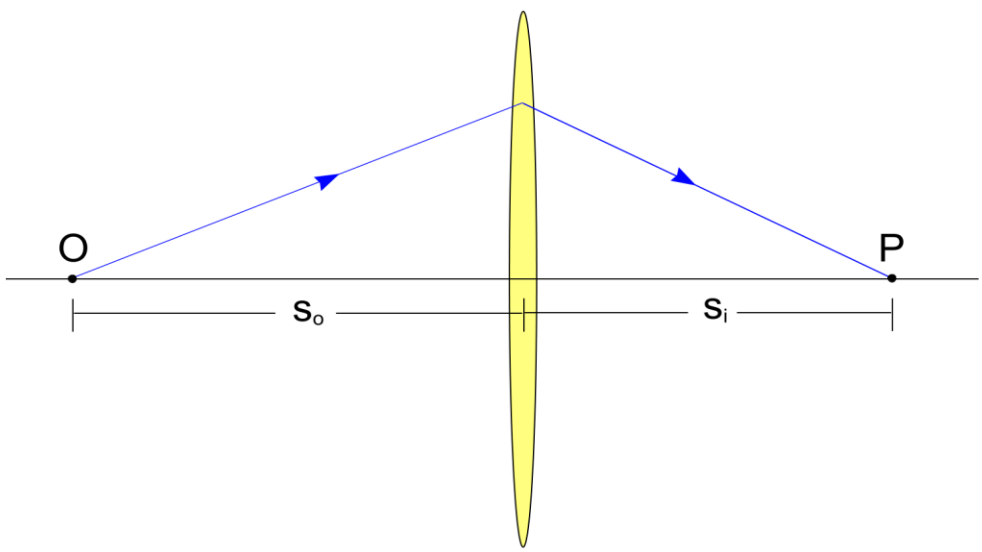
V1. Fresnel Zone Plates. Recall from last class our wavefront and Fresnel zones.



Comparing the two figures here, we see that the wavefront of Fresnel zones acts like a lens.



Recall the formula for a lens: $$\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f}$$

Identify s_o with z' and s_i with z.

Then $$\frac{1}{z'} + \frac{1}{z} = \frac{1}{f}.$$

Also from last class: $\Delta_m = \frac{R_m^2}{2}(\frac{1}{z'} + \frac{1}{z}) = m\frac{\lambda}{2}$ and we defined

$\frac{1}{L} \equiv \frac{1}{z'} + \frac{1}{z}$, which led to $\frac{R_m^2}{2}\frac{1}{L} = m\frac{\lambda}{2}$ and $R_m^2 = m\lambda L$.

Now we can identify L as the focal length.

$$f = L$$

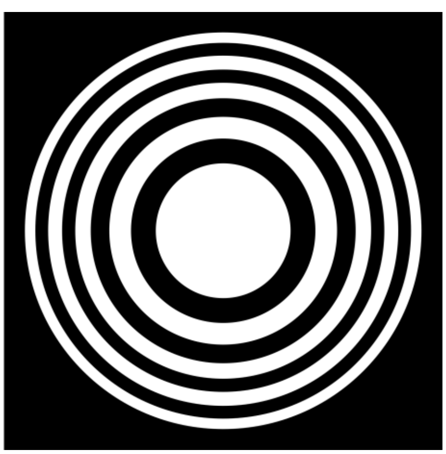
Note that the focal length is independent of any particular Fresnel zone.

Since $R_m^2 = m\lambda L$, we can also write the focal length as

$$f = \frac{R_m^2}{m\lambda}.$$

We can pick any m since f = L = constant. So pick m = 1 and we have

$$f = \frac{R_1^2}{\lambda}.$$



Immediately evident is the fact that the focal length depends strongly on wavelength, giving much chromatic aberration. The Fresnel zone plate is a filter with concentric rings that block some Fresnel zones. In the figure some rings are transparent (white), letting light through, and some rings are opaque (black), blocking light.

Such plates are named in honor of Fresnel.

Fresnel Zone Plate
Wikipedia: Tom Murphy VII. Creative Commons

The zone plate focuses light by diffraction, while a lens focuses light by refraction. If you block out the even Fresnel zones, you will get a strong focus as the odd zones reinforce each other.

Engineering Design Example 1. Consider a Fresnel zone plate with 10 concentric rings where the odd zones can pass through and the even zones are blocked. The plate is designed so that 500-nm light is focused at a distance 50 cm from the plate. Let's consider the incoming light as plane waves where the object distance can be taken to be infinity. Then, the image distance $s_i = f$.

The radius of the first zone is found from $f = \frac{R_1^2}{\lambda}$. The radius is

$$R_1 = \sqrt{\lambda f} = \sqrt{500\times10^{-9}\cdot 50\times10^{-2}} \text{ m}$$

$$R_1 = \sqrt{500\times10^{-9}\cdot 5\times10^{-1}} \quad \Rightarrow \quad R_1 = \sqrt{2500\times10^{-10}} = 50\times10^{-5}$$

$$R_1 = 0.50\times10^{-3} = 0.5 \text{ mm}$$

Knocking out 10 even zones means that the field strength is enhanced about 10 times.

The irradiance is the approximately 100 times stronger.

The approximation is rough and ignores the obliquity factor.

When we loosely say brighter, we mean the irradiance as measured by a scientific instrument.

Visual perception in nonlinear and complicated.

Something with twice the irradiance will not appear exactly twice as bright to the eye.

Engineering Design Example 2. Consider a zone plate where $R_1 = 1 \text{ cm}$. For this design, what is the focal length for 500-nm light?

$$f = \frac{R_1^2}{\lambda} = \frac{(0.01 \text{ m})^2}{500\times10^{-9} \text{ m}} = \frac{1}{5}\frac{10^{-4}}{10^{-7}} = \frac{10^3}{5} = \frac{1000}{5} = 200 \text{ m}$$

That is about two US football fields, a "lens" equivalent of $\frac{1}{200} = 0.005 \text{ diopters}$.

Summary: $f = \frac{R_1^2}{\lambda}$ and the corresponding diopter strength is $\frac{1}{f} = \frac{\lambda}{R_1^2}$.

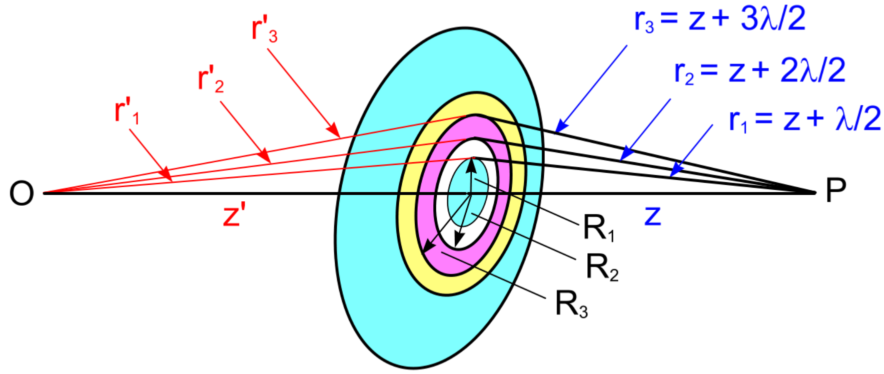
V2. The Fresnel Approximation. Below is an unobstructed wave front and then an opening cut-out. Make the following changes in variable notation:

$r' \rightarrow \rho$,

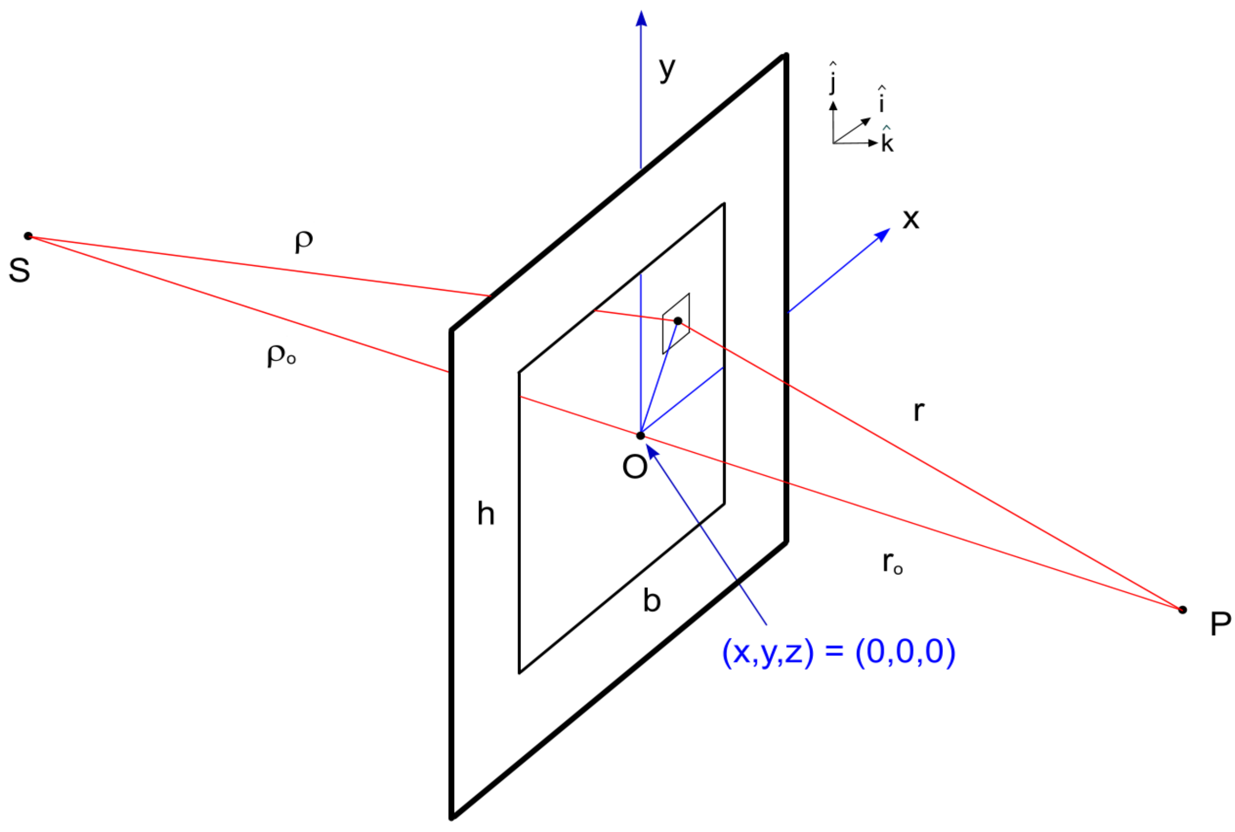
$z' \rightarrow \rho_o$

$z \rightarrow r_o$,

$r \rightarrow r$ (no change).



Notation is easier when we have to square things without primes. We have the setup below



with the corresponding formula from last class

$$E_p = \frac{E_o}{zz'} \int_A e^{ik(r'+r)}\, dA \rightarrow \frac{E_o}{\rho_o r_o} \int_A e^{ik(\rho+r)}\, dA .$$

Now we make the Fresnel approximation where x and y are much smaller than ρ_o and r_o.

$$\rho = (\rho_o^2 + x^2 + y^2)^{1/2} = \rho_o(1 + \frac{x^2}{\rho_o^2} + \frac{y^2}{\rho_o^2})^{1/2}$$

$$\rho = (\rho_o^2 + x^2 + y^2)^{1/2} = \rho_o(1 + \frac{1}{2}\frac{x^2}{\rho_o^2} + \frac{1}{2}\frac{y^2}{\rho_o^2}) \approx \rho_o + \frac{x^2}{2\rho_o} + \frac{y^2}{2\rho_o}$$

Similarly, $r = (r_o^2 + x^2 + y^2)^{1/2} \approx r_o + \frac{x^2}{2r_o} + \frac{y^2}{2r_o}$.

Then $\rho + r \approx \rho_o + \frac{x^2}{2\rho_o} + \frac{y^2}{2\rho_o} + r_o + \frac{x^2}{2r_o} + \frac{y^2}{2r_o}$.

$$\rho + r \approx \rho_o + r_o + \frac{(x^2 + y^2)}{2}(\frac{1}{\rho_o} + \frac{1}{r_o})$$

$$\rho + r \approx \rho_o + r_o + \frac{(x^2 + y^2)}{2}(\frac{\rho_o + r_o}{\rho_o r_o})$$

$$E_p = \frac{E_o}{\rho_o r_o}\int_A e^{ik(\rho + r)}\, dA = \frac{E_o}{\rho_o r_o}\int_A e^{ik[\rho_o + r_o + \frac{(x^2+y^2)}{2}(\frac{\rho_o + r_o}{\rho_o r_o})]}\, dA$$

$$E_p = \frac{E_o}{\rho_o r_o}\int_A e^{ik(\rho + r)}\, dA = \frac{E_o}{\rho_o r_o} e^{ik(\rho_o + r_o)}\int_A e^{ik\frac{(x^2+y^2)}{2}(\frac{\rho_o + r_o}{\rho_o r_o})}\, dA$$

$E_p = C\int_A e^{ik\frac{(x^2+y^2)}{2}(\frac{\rho_o + r_o}{\rho_o r_o})}\, dA$ where C is a constant.

$$E_p = C\int_{x_1}^{x_2}\int_{y_1}^{y_2} e^{ik\frac{(x^2+y^2)}{2}(\frac{\rho_o+r_o}{\rho_o r_o})}dxdy$$

Here is the x^2 term hitting the "i" in the exponent: $k\frac{x^2}{2}(\frac{\rho_o+r_o}{\rho_o r_o})$.

Use $k=\frac{2\pi}{\lambda}$ and we have $\frac{2\pi}{\lambda}\frac{x^2}{2}(\frac{\rho_o+r_o}{\rho_o r_o}) = \frac{\pi}{2}x^2\left[\frac{2(\rho_o+r_o)}{\lambda\rho_o r_o}\right]$.

Define $u = x\left[\frac{2(\rho_o+r_o)}{\lambda\rho_o r_o}\right]^{1/2}$ and $v = y\left[\frac{2(\rho_o+r_o)}{\lambda\rho_o r_o}\right]^{1/2}$

so that the exponent looks cute: $e^{i\pi(u^2+v^2)/2} = e^{i\pi u^2/2}e^{i\pi v^2/2}$.

The differentials are $du = \left[\frac{2(\rho_o+r_o)}{\lambda\rho_o r_o}\right]^{1/2} dx$ and $dv = \left[\frac{2(\rho_o+r_o)}{\lambda\rho_o r_o}\right]^{1/2} dy$.

$$E_p = C\int_{x_1}^{x_2}\int_{y_1}^{y_2} e^{ik(\rho+r)}dxdy \rightarrow$$

$$C\int_{u_1}^{u_2}\int_{v_1}^{v_2} e^{i\pi u^2/2}e^{i\pi v^2/2}\frac{du}{\left[\frac{2(\rho_o+r_o)}{\lambda\rho_o r_o}\right]^{1/2}}\frac{dv}{\left[\frac{2(\rho_o+r_o)}{\lambda\rho_o r_o}\right]^{1/2}}$$

$$E_p = C\frac{\lambda\rho_o r_o}{2(\rho_o+r_o)}\int_{u_1}^{u_2}\int_{v_1}^{v_2} e^{i\pi u^2/2}e^{i\pi v^2/2}dudv$$

$E_p = C'\int_{u_1}^{u_2} e^{i\pi u^2/2}du\int_{v_1}^{v_2} e^{i\pi v^2/2}dv$, where C' is a new constant.

V3. The Fresnel Integrals.

Integrals like $\int e^{i\pi u^2/2} du = \int \left[\cos(\frac{\pi u^2}{2}) + i\sin(\frac{\pi u^2}{2}) \right] du$ naturally appear.

The following are called the Fresnel integrals:

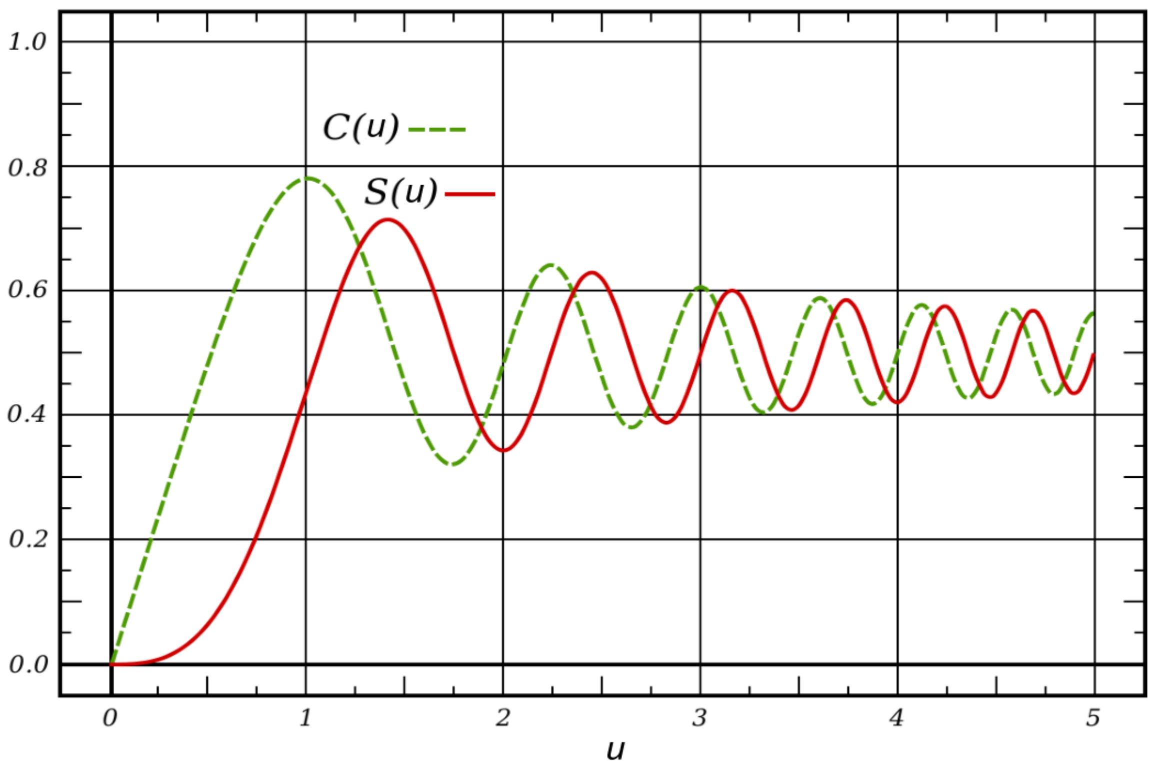
$$C(u) \equiv \int_0^u \cos(\frac{\pi u^2}{2}) du \qquad S(u) \equiv \int_0^u \sin(\frac{\pi u^2}{2}) du$$

Don't worry about using u as an integration variable and as an integration limit. A common example of such usage in physics is deriving kinetic energy from work.

$$W = \int F dx = \int m a dx = \int m \frac{dv}{dt} dx = m \int v dv$$

$$W = m \int_0^v v dv = m \left. \frac{v^2}{2} \right|_0^v = \frac{1}{2} m v^2$$

Fresnel Integrals C(u) and S(u)



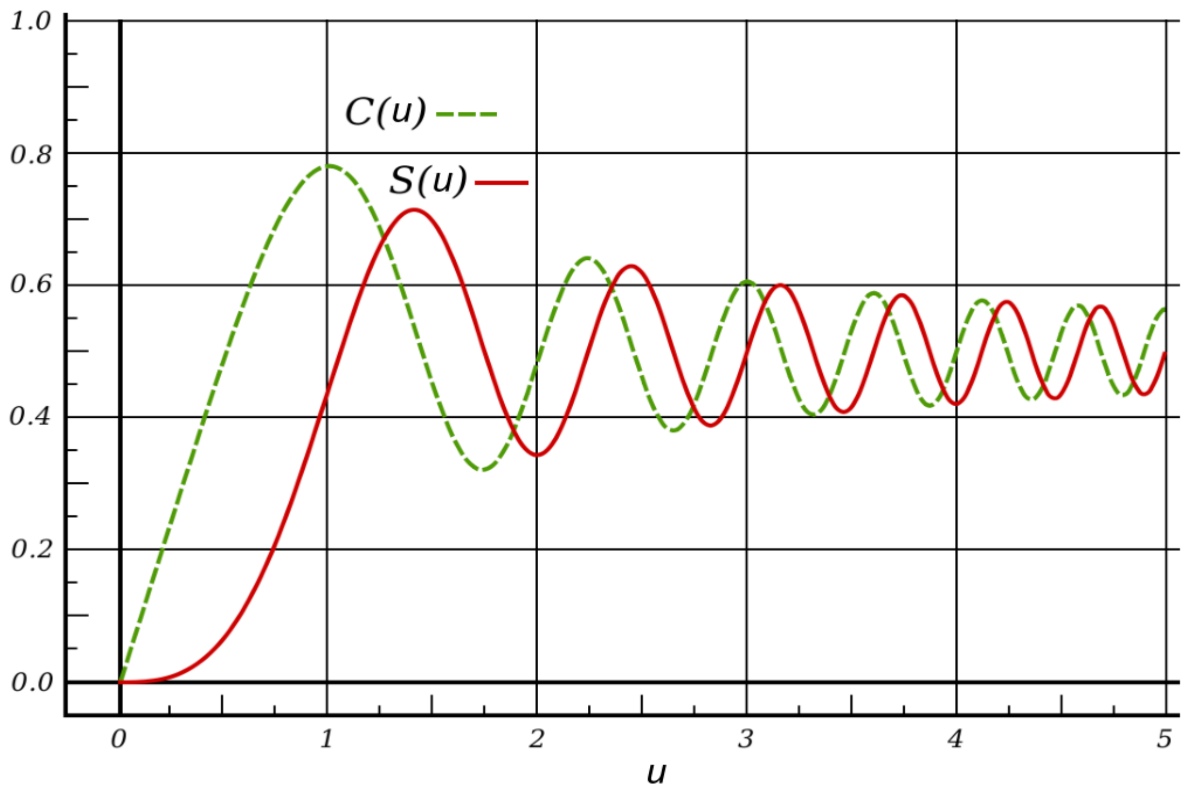
Wikipedia: Inductiveload, using Mathematica and Inkscape. Public Domain.

Let's see if we can understand the integral results for small u.

For $u \approx 0$, we have $C(u) = \int_0^u \cos(\frac{\pi u^2}{2}) du \approx \int_0^u (1) du = u$.

See dashed green straight-line section in the above figure for small u.

The slope of that dashed line is 1. Extend it up and you will hit the point (1,1).



For $u \approx 0$, $S(u) \equiv \int_0^u \sin(\frac{\pi u^2}{2}) du \approx \int_0^u \frac{\pi u^2}{2} du = \frac{\pi u^3}{2 \cdot 3} \approx \frac{u^3}{2}$.

See this result in the rising red curve for small u.

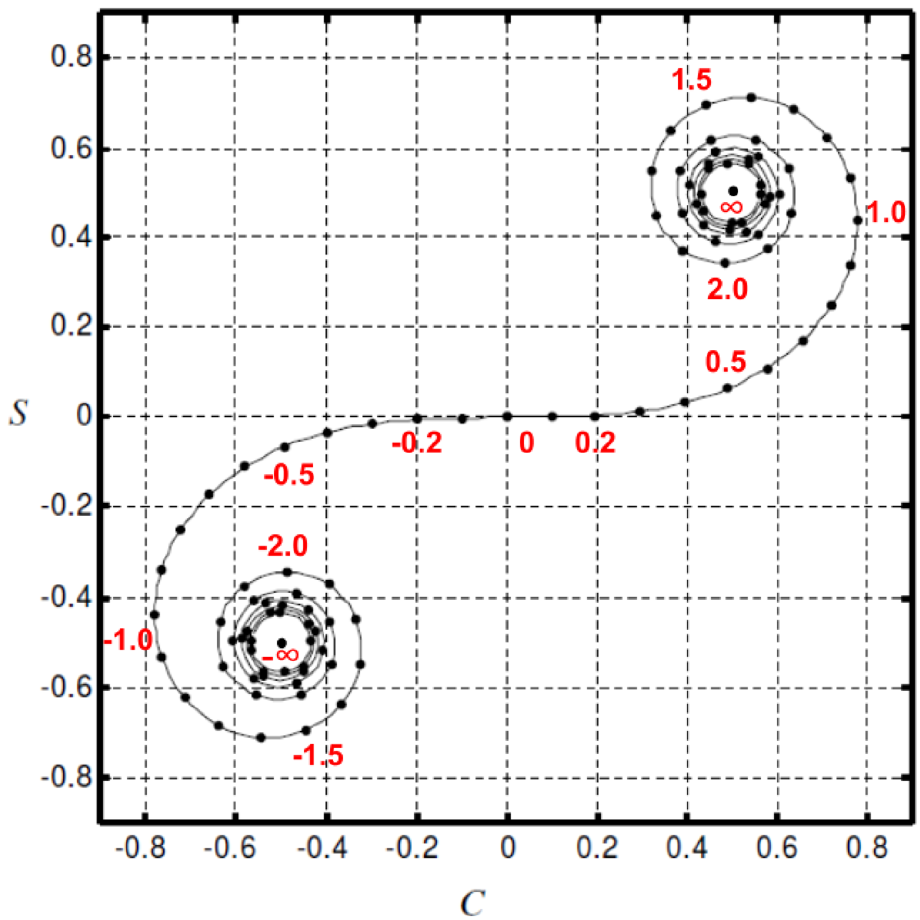
In particular, note $\left.\frac{u^3}{2}\right|_{u=1} = \frac{1}{2} = 0.5$. The red graph passes near (1, 0.5).

Note that $[C(0), S(0)] = [0, 0]$ and $[C(1), S(1)] \approx [0.8, 0.5]$.

V4. The Cornu Spiral. Here we plot C(u) + i S(u) in the complex plane.

$$C(u) \equiv \int_0^u \cos(\frac{\pi u^2}{2})du \qquad S(u) \equiv \int_0^u \sin(\frac{\pi u^2}{2})du$$

A parametric plot $[C(u), S(u)]$ gives the famous Euler or Cornu spiral.



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The labels along the spiral are the u values themselves! Watch!

$$ds^2 = dC^2 + dS^2 = \cos(\frac{\pi u^2}{2})du^2 + \sin(\frac{\pi u^2}{2})du^2 = du^2$$

$$ds = du$$

The arc length is ds = du. Values of u are labeled along the spiral. Do they match?

Observation 1. Note that $[C(0), S(0)] = [0, 0]$ is at the center of the plot.

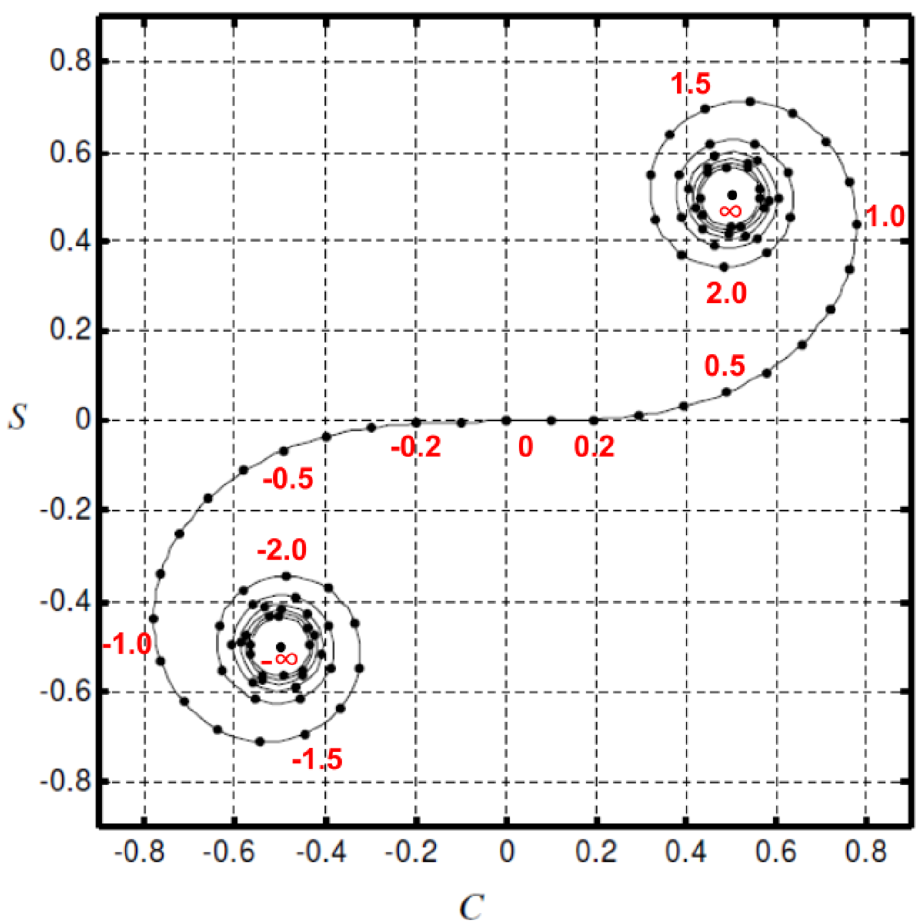
Observation 2. Note that $[C(1), S(1)] \approx [0.8, 0.5]$ is positioned correctly.

Observation 3. Note u values along the spiral match the axes values by focusing your attention on the horizontal labels from -0.2 to +0.2.

Observation 4. Note that for small u our $[C(u), S(u)] \approx [u, \frac{u^3}{2}]$ are seen in the plot.

Observation 5. Note how the decreasing oscillations in $C(u)$ and $S(u)$ from the graphs on the previous page lead to the shrinking spiral shape in the upper right section.

Observation 6. Note that $C(u)$ is an even function since we are dealing with a cosine, while $S(u)$ is odd due to the sine. Therefore, in negative space there is another spiral.



Observation 7. The slope.

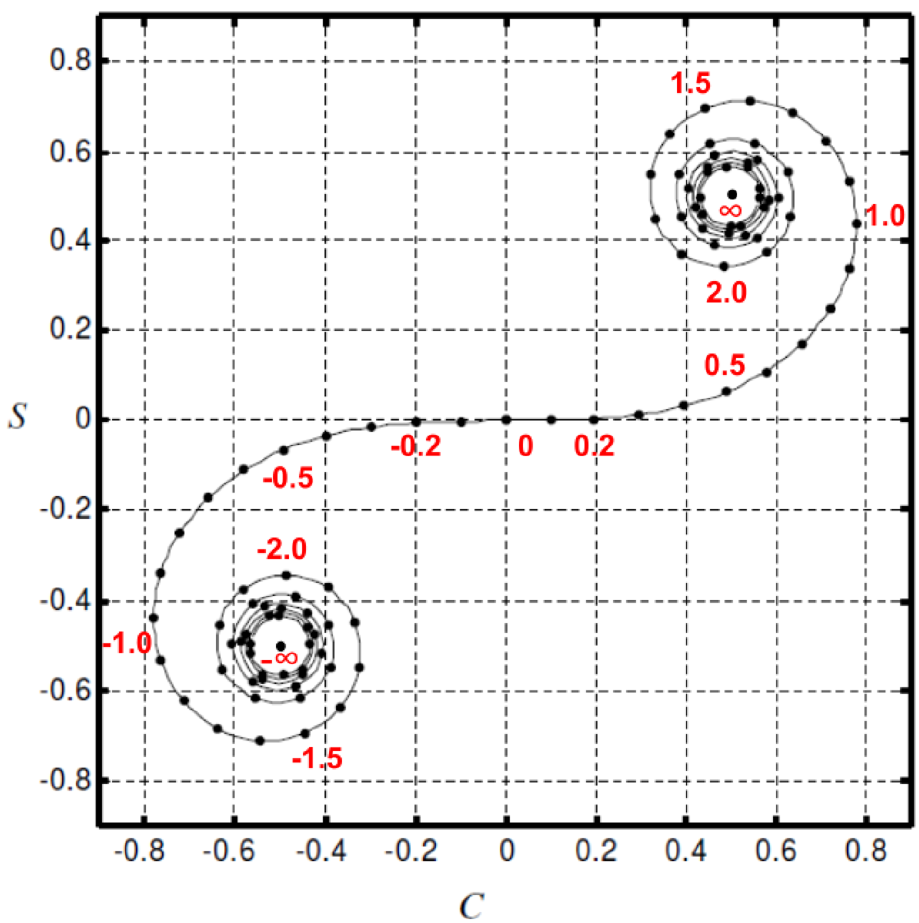
$$C(u) = \int_0^u \cos(\frac{\pi u^2}{2})du \qquad S(u) = \int_0^u \sin(\frac{\pi u^2}{2})du$$

$$dC(u) = \cos(\frac{\pi u^2}{2})du \qquad dS(u) = \sin(\frac{\pi u^2}{2})du$$

$$\frac{dS(u)}{dC(u)} = \tan(\frac{\pi u^2}{2})du$$

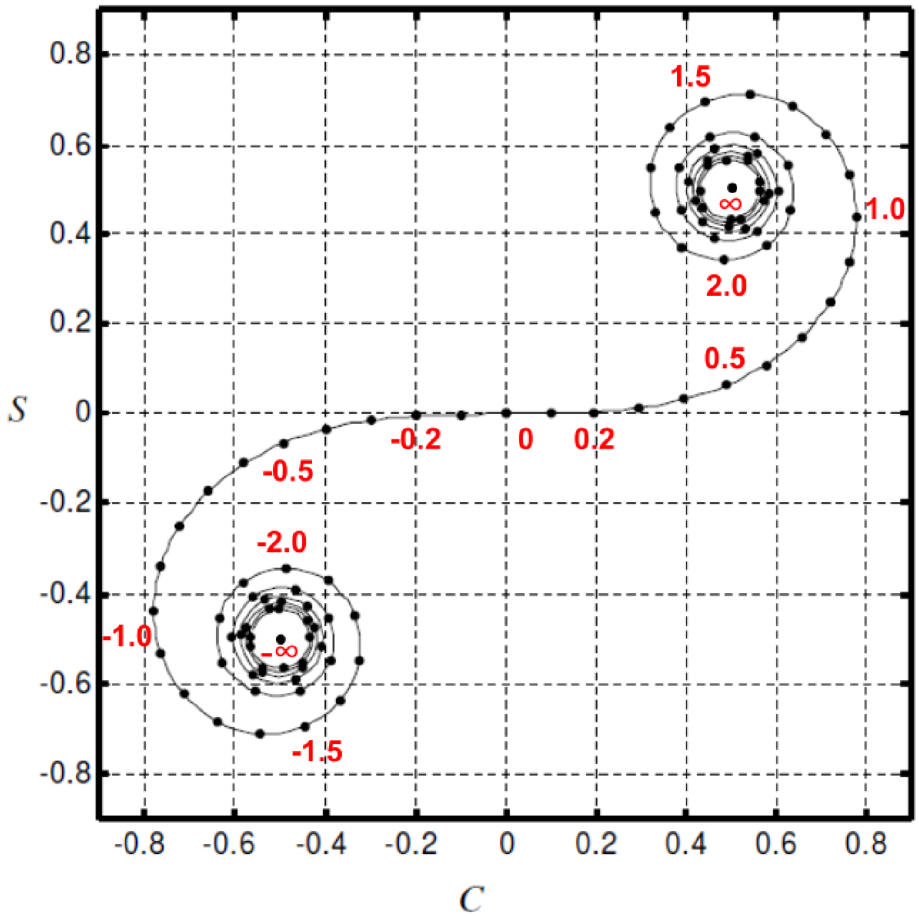
Slope is zero when $\sin(\frac{\pi u^2}{2}) = 0$, i.e., $\frac{\pi u^2}{2} = 0, \pi, 2\pi, \ldots$.

$$u^2 = 0, 2, 4, \ldots \quad => \quad u = 0, \pm\sqrt{2}, \pm 2, \ldots$$



Slope is infinite when $\cos(\frac{\pi u^2}{2}) = 0$, i.e., $\frac{\pi u^2}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$.

$u^2 = 1, 3, 5, \ldots$ => $u = \pm 1, \pm\sqrt{3}, \pm\sqrt{5}, \ldots$



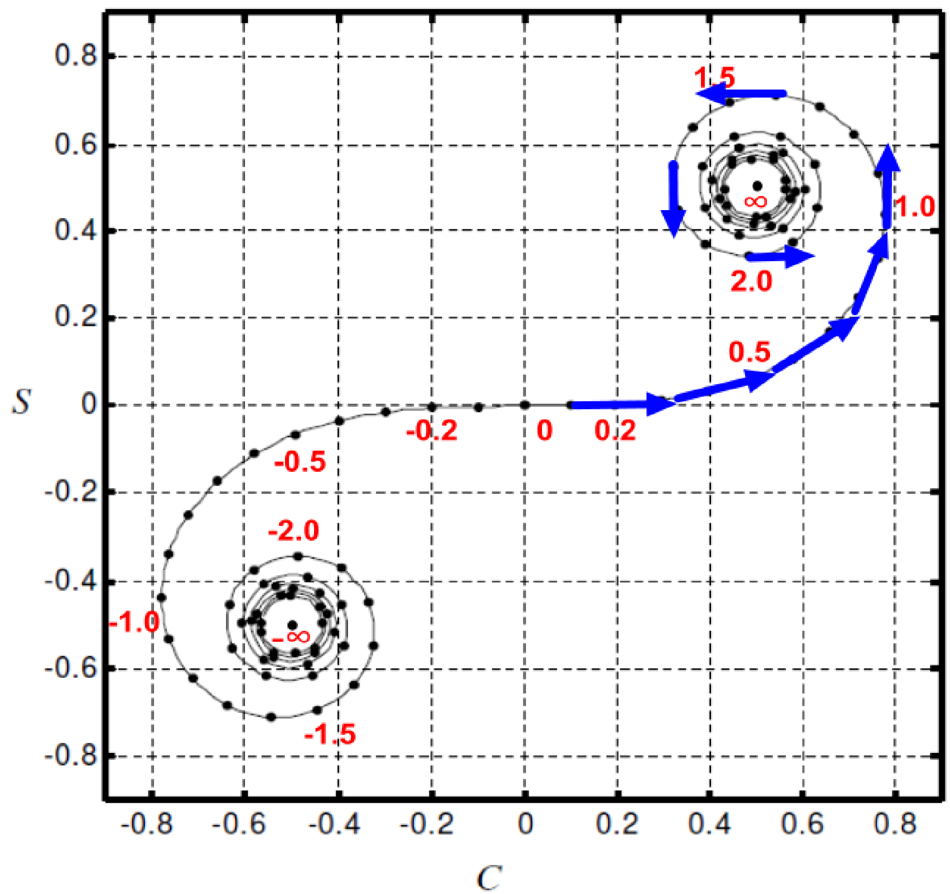
Wasn't the numerical analysis easier because of the $\frac{\pi u^2}{2}$ as the argument for the cosine and sine? It makes one appreciate the convention for the u and v variables earlier that led to

$$C(u) \equiv \int_0^u \cos(\frac{\pi u^2}{2}) du \quad \text{and} \quad S(u) \equiv \int_0^u \sin(\frac{\pi u^2}{2}) du .$$

Observation 8. A Vibration Curve. Baby Phasors.

Remember from calculus when you have an integral $\int f(x)dx$ that you set up things so that you have strips of area $\sum f(x)\Delta x$. We have a similar idea here with the phase integration.

$$\int e^{i\pi u^2/2}du \rightarrow \sum e^{i\pi u^2/2}\Delta u$$



V5. Using Cornu Spiral. Here is how to use the Cornu spiral.

$$\int_{u_1}^{u_2} e^{i\pi u^2/2} du = \int_{u_1}^{0} e^{i\pi u^2/2} du + \int_{0}^{u_2} e^{i\pi u^2/2} du$$

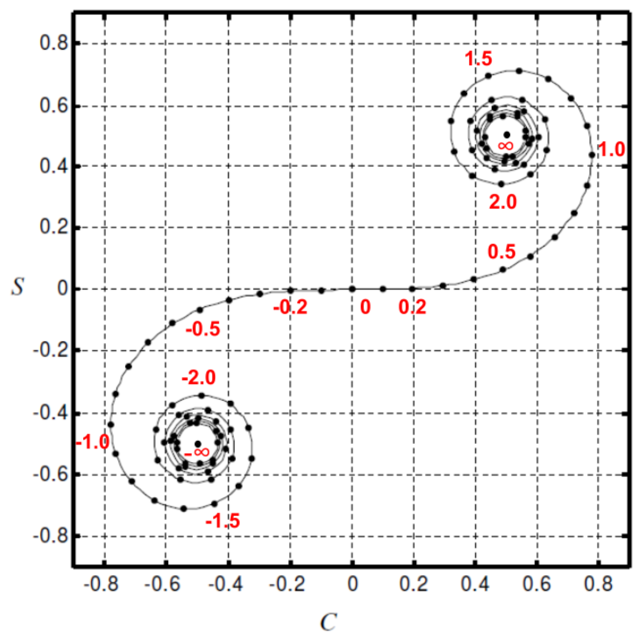
$$\int_{u_1}^{u_2} e^{i\pi u^2/2} du = -\int_{0}^{u_1} e^{i\pi u^2/2} du + \int_{0}^{u_2} e^{i\pi u^2/2} du$$

$$\int_{u_1}^{u_2} e^{i\pi u^2/2} du = \int_{0}^{u_2} e^{i\pi u^2/2} du - \int_{0}^{u_1} e^{i\pi u^2/2} du$$

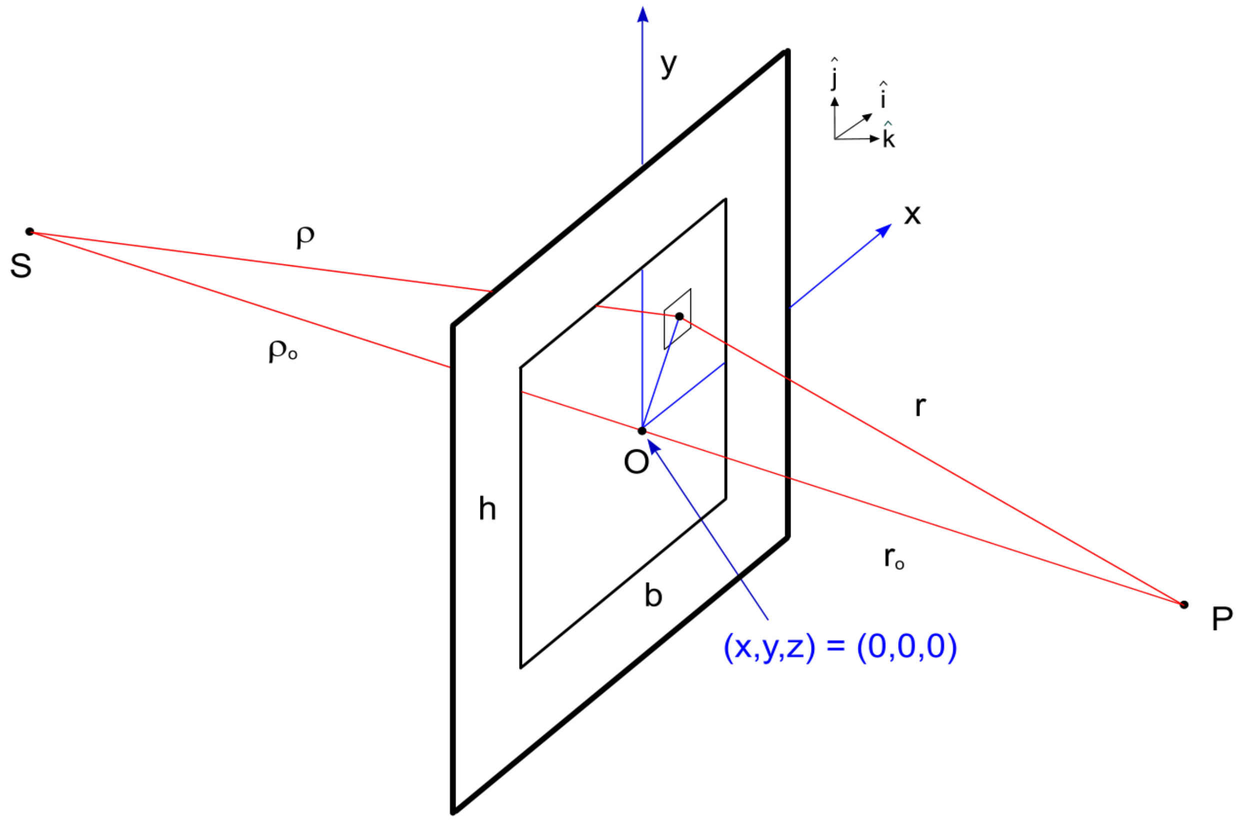
$$\int_{u_1}^{u_2} e^{i\pi u^2/2} du = [C(u_2) + iS(u_2)] - [C(u_1) - iS(u_1)]$$

$$\int_{u_1}^{u_2} e^{i\pi u^2/2} du = [C(u) + iS(u)]\Big|_{u_1}^{u_2}$$

Consulting the Cornu spiral does the integral for us.



V6. The Unobstructed Beam. We return now to our aperture setup from before.



$$E_p = \frac{A_o}{2}\int_{u_1}^{u_2} e^{i\pi u^2/2}du \int_{v_1}^{v_2} e^{i\pi v^2/2}dv$$

$$u = x\left[\frac{2(\rho_o + r_o)}{\lambda \rho_o r_o}\right]^{1/2} \quad \text{and} \quad v = y\left[\frac{2(\rho_o + r_o)}{\lambda \rho_o r_o}\right]^{1/2}$$

For the unobstructed beam the rectangular aperture below enlarges to infinity in the x and y directions. As x and y expand to infinity in each direction so do u and v. Therefore, we want

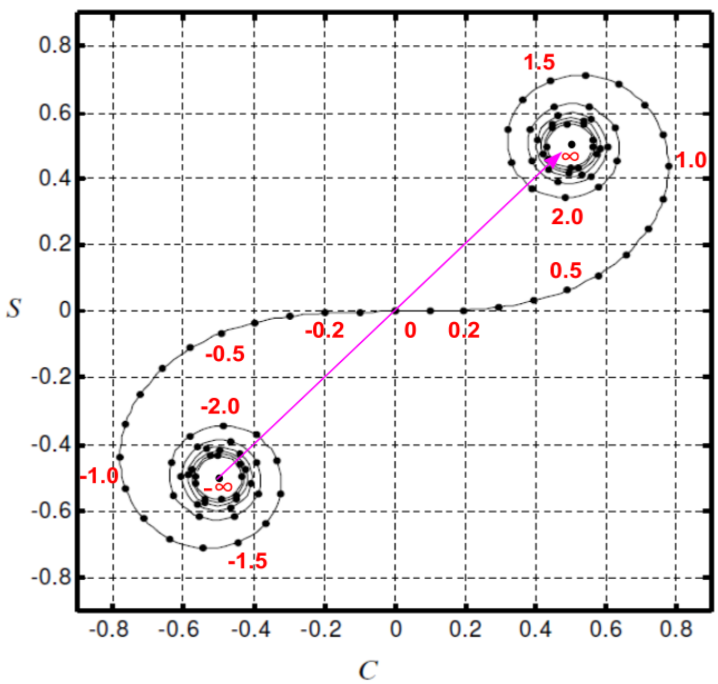
$$E_p = E_o\int_{-\infty}^{\infty} e^{i\pi u^2/2}du \int_{-\infty}^{\infty} e^{i\pi v^2/2}dv$$, redefining the constant in front.

Then aperture has enlarged to infinity. There is no aperture any more.

Let's focus on one of the integrals. $\int_{-\infty}^{\infty} e^{i\pi u^2/2} du = C(u)\Big|_{-\infty}^{\infty} + i\,S(u)\Big|_{-\infty}^{\infty}$

$$\int_{-\infty}^{\infty} e^{i\pi u^2/2} du = [C(\infty) - C(-\infty)] + i[S(\infty) - S(-\infty)] = a + bi$$

$$\left| \int_{-\infty}^{\infty} e^{i\pi u^2/2} du \right| = \sqrt{a^2 + b^2}$$



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From the Cornu spiral

$$C(\infty) = 0.5 \qquad S(\infty) = 0.5$$
$$C(-\infty) = -0.5$$
$$S(-\infty) = -0.5$$
$$\int_{-\infty}^{\infty} e^{i\pi u^2/2} du = 1 + i$$

You can just look at the total length of the connecting line.

$$E_p = E_o \int_{u_1}^{u_2} e^{i\pi u^2/2} du \int_{v_1}^{v_2} e^{i\pi v^2/2} dv = E_o (1+i)(1+i)$$

$$\left|E_p\right|^2 = E_o^2 (1+i)(1+i)(1-i)(1-i) = E_o^2 (1+i)(1-i)(1+i)(1-i)$$

$$\left|E_p\right|^2 = E_o^2 (1+1)(1+1) = E_o^2 (2)(2) = 4E_o^2 \equiv A_o^2$$

Normalization for the unobstructed case: $E_p = \dfrac{A_o}{2} \int_{u_1}^{u_2} e^{i\pi u^2/2} du \int_{v_1}^{v_2} e^{i\pi v^2/2} dv$.

V7. Phasor Properties. We will be examining in the next chapter a long narrow slit. For such a slit, one of the integrals will be from negative infinity to plus infinity.

$$E_p = \frac{A_o}{2} \int_{u_1}^{u_2} e^{i\pi u^2/2} du \int_{v_1}^{v_2} e^{i\pi v^2/2} dv = \frac{A_o}{2} (1+i) \int_{v_1}^{v_2} e^{i\pi v^2/2} dv$$

Then we will want

$$\left| E_p \right| = \left| \frac{A_o}{2} (1+i) \int_{v_1}^{v_2} e^{i\pi v^2/2} dv \right| .$$

We now will address two complex numbers or phasors, where

$A = a + bi$ and $B = c + di$

What is the modulus $|uv|$?

$$AB = (a+bi)(c+di) = ac + adi + bic - bd$$

$$AB = ac - bd + i(ad + bc)$$

$$|AB| = \sqrt{(ac-bd)^2 + (ad+bc)^2}$$

$$|AB| = \sqrt{a^2c^2 - 2acbd + b^2d^2 + a^2d^2 + 2adbc + b^2c^2}$$

$$|AB| = \sqrt{a^2(c^2+d^2) + b^2(c^2+d^2)}$$

$$|AB| = \sqrt{(a^2+b^2)(c^2+d^2)} = |A||B|$$

$$|AB| = |A||B|$$

Therefore, $\left|E_p\right| = \left|\frac{A_o}{2}(1+i)\right| \left|\int_{v_1}^{v_2} e^{i\pi v^2/2} dv\right| = \frac{A_o}{\sqrt{2}} \left|\int_{v_1}^{v_2} e^{i\pi v^2/2} dv\right|$.

So, we can write with no problem for a one-dimensional slit

$E_p = \frac{A_o}{\sqrt{2}} \int_{v_1}^{v_2} e^{i\pi v^2/2} dv$ or we can use $E_p = \frac{A_o}{\sqrt{2}} \int_{u_1}^{u_2} e^{i\pi u^2/2} du$.

The Euler or Cornu Spiral

 Leonhard Euler	 Alfred Cornu
1707 – 1783	1841 - 1902
Swiss Mathematician, Physicist Astronomer, Engineer	French Physicist

James Bernoulli studied the curve where arc length advances linearly according to the radius of curvature at each point. Euler developed the mathematics for the curve. The spiral is named after him as well as Cornu, the latter working in optics.

The Cornu spiral is used in road design such as the cloverlead interchange.

Here is another illustration of the power and breadth of physics.

Your study of the intricacies of optics easily translates to studies in engineering.

Spirals Allow for the Safest Driving Along Entrances and Exits of Highways

Wikipedia: "Typical cloverleaf interchange with collector/distributor roads in Wyoming, Michigan." Courtesy the Michigan Department of Transportation.
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