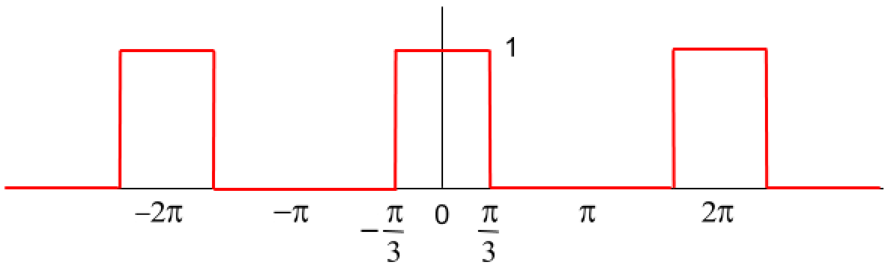
Modern Optics, Prof. Ruiz, UNCA doctorphys.com
Chapter R. Fourier

HW R1. Fourier Series.

Find the Fourier Series for the periodic wave shown below. Your basic cycle for this repeating pattern is defined over our standard region $-\pi \le x \le \pi$, symmetric about the origin, where the pulse is 1/3 the period (or wavelength) of the periodic wave.



Pulse Train with Duty Cycle $33\frac{1}{3}\%$

For full credit, write out your answer by giving the first six nonzero terms. You must write f(x) = and then give the coefficients multiplied by the appropriate trig function for 6 nonzero terms where each coefficient is in simplest non-decimal mathematical form.

Square Pulse Train with $33\frac{1}{3}\%$ duty cycle with pulse center on zero. This means an even function so that we only have the constant and the cosines.

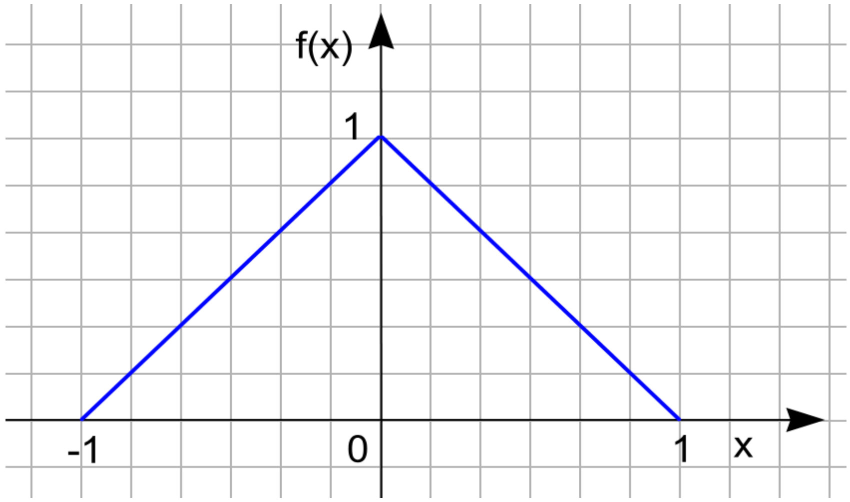
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi/3} 1\cdot dx = \frac{2}{\pi}\,x\Big|_0^{\pi/3} = \frac{2}{\pi}\frac{\pi}{3} = \frac{2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi/3}\cos(nx)\,dx = \frac{2}{\pi}\frac{\sin(nx)}{n}\Bigg|_0^{\pi/3} = \frac{2}{\pi}\frac{\sin(n\pi/3)}{n}$$

For n = 1, 2, 3, 4, 5, 6, 7, ... $\sin(n\pi/3)$ gives $\frac{\sqrt{3}}{2}$, $\frac{\sqrt{3}}{2}$, 0, $-\frac{\sqrt{3}}{2}$, $-\frac{\sqrt{3}}{2}$, 0, $\frac{\sqrt{3}}{2}$, ...

$$f(x) = \frac{1}{3} + \frac{\sqrt{3}}{\pi}\left[\cos x + \frac{\cos 2x}{2} - \frac{\cos 4x}{4} - \frac{\cos 5x}{5} + \frac{\cos 7x}{7} + \ldots\right]$$

HW R2. Fourier Transform.



$$\Im\{f(x)\} \equiv F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \left[\int_{-1}^{0} (1+x) e^{-ikx} dx + \int_{0}^{1} (1-x) e^{-ikx} dx \right]$$

$$\sqrt{2\pi} F(k) = \int_{-1}^{0} e^{-ikx} dx + \int_{-1}^{0} x e^{-ikx} dx + \int_{0}^{1} e^{-ikx} dx - \int_{0}^{1} x e^{-ikx} dx$$

$$\sqrt{2\pi} F(k) \equiv I_1 + I_2 + I_3 + I_4$$

$$I_1 = \int_{-1}^{0} e^{-ikx} dx = \left. \frac{e^{-ikx}}{-ik} \right|_{-1}^{0} = \frac{e^0 - e^{ik}}{-ik} = \frac{1 - e^{ik}}{-ik} = \frac{e^{ik} - 1}{ik}$$

$$I_2 = \int_{-1}^{0} x e^{-ikx} dx = i \int_{-1}^{0} \frac{d}{dk} (e^{-ikx}) dx = i \frac{d}{dk} \left(\frac{e^{ik} - 1}{ik} \right)$$

$$I_2 = \frac{d}{dk}\left(\frac{e^{ik}-1}{k}\right) = \frac{1}{k}(i)e^{ik} + (e^{ik}-1)\left(-\frac{1}{k^2}\right)$$

$$\boxed{I_1 = \int_{-1}^{0} e^{-ikx}dx = \frac{e^{ik}-1}{ik}}$$

$$\boxed{I_2 = \int_{-1}^{0} xe^{-ikx}dx = \frac{i}{k}e^{ik} + \frac{1}{k^2}(1-e^{ik})}$$

$$I_3 = \int_0^1 e^{-ikx}dx = \left.\frac{e^{-ikx}}{-ik}\right|_0^1 = \frac{e^{-ik}-e^0}{-ik} = \frac{e^{-ik}-1}{-ik} = \frac{1-e^{-ik}}{ik}$$

$$I_4 = -\int_0^1 xe^{-ikx}dx = -i\int_0^1 \frac{d}{dk}(e^{-ikx})dx = -i\,\frac{d}{dk}\left(\frac{1-e^{-ik}}{ik}\right)$$

$$I_4 = \frac{d}{dk}\left(\frac{e^{-ik}-1}{k}\right) = \frac{1}{k}(-i)e^{-ik} + (e^{-ik}-1)\left(-\frac{1}{k^2}\right)$$

$$\boxed{I_3 = \int_0^1 e^{-ikx}dx = \frac{1-e^{-ik}}{ik}}$$

$$\boxed{I_4 = -\int_0^1 xe^{-ikx}dx = -\frac{i}{k}e^{-ik} + \frac{1}{k^2}(1-e^{-ik})}$$

$$\sqrt{2\pi}F(k) \equiv I_1 + I_2 + I_3 + I_4$$

$$I_1 + I_3 = \frac{e^{ik} - 1}{ik} + \frac{1 - e^{-ik}}{ik} = \frac{e^{ik} - e^{-ik}}{ik}$$

$$I_2 + I_4 = \frac{i}{k} e^{ik} + \frac{1}{k^2}(1 - e^{ik}) - \frac{i}{k} e^{-ik} + \frac{1}{k^2}(1 - e^{-ik})$$

$$I_2 + I_4 = \frac{i}{k}(e^{ik} - e^{-ik}) + \frac{1}{k^2}(1 - e^{ik} + 1 - e^{-ik})$$

$$I_1 + I_2 + I_3 + I_4 = \frac{e^{ik} - e^{-ik}}{ik} + \frac{i}{k}(e^{ik} - e^{-ik}) + \frac{1}{k^2}(1 - e^{ik} + 1 - e^{-ik})$$

$$I_1 + I_2 + I_3 + I_4 = \frac{e^{ik} - e^{-ik}}{ik} - \frac{e^{ik} - e^{-ik}}{ik} + \frac{1}{k^2}(1 - e^{ik} + 1 - e^{-ik})$$

$$\sqrt{2\pi} F(k) = I_1 + I_2 + I_3 + I_4 = \frac{1}{k^2}(1 - e^{ik} + 1 - e^{-ik})$$

$$\sqrt{2\pi} F(k) = \frac{1}{k^2}(1 - e^{ik} + 1 - e^{-ik})$$

$$\sqrt{2\pi} F(k) = -\frac{1}{k^2}(e^{ik} - 2 + e^{-ik})$$

$$\sqrt{2\pi} F(k) = -\frac{1}{k^2}(e^{ik/2} - e^{-ik/2})^2$$

$$\sqrt{2\pi} F(k) = \frac{1}{k^2} \frac{(e^{ik/2} - e^{-ik/2})^2}{(-1)} = \frac{1}{k^2} \frac{(e^{ik/2} - e^{-ik/2})^2}{i^2}$$

$$\sqrt{2\pi}F(k) = \frac{1}{k^2}\frac{(e^{ik/2} - e^{-ik/2})^2}{(-1)} = \frac{4}{k^2}\left[\frac{e^{ik/2} - e^{-ik/2}}{2i}\right]^2$$

$$\sqrt{2\pi}F(k) = \frac{4}{k^2}\sin^2(k/2) = \frac{\sin^2(k/2)}{(k/2)^2} = \left[\frac{\sin(k/2)}{(k/2)}\right]^2$$

$$\sqrt{2\pi}F(k) = \text{sinc}^2(k/2)$$

$$\boxed{F(k) = \frac{\text{sinc}^2(k/2)}{\sqrt{2\pi}}}$$