HW O1. Elliptical Polarization. Consider the electric wave traveling down the z-axis with the most general polarization state

$$\vec{E} = E_x \hat{i} + E_y \hat{j} = E_{ox} \hat{i} \cos(kz - \omega t) + E_{oy} \hat{j} \cos(kz - \omega t + \phi) .$$

Here you will take $\phi = \dfrac{\pi}{2}$ and $E_{ox} > E_{oy}$. Set $z = 0$ and let $\theta = -\omega t$ like we did in class and show that the electric field vector traces out an ellipse. Give the equation for the ellipse in terms of the variables E_x, E_y, and constants E_{ox}, E_{oy}. Finally give the eccentricity in terms of the constants. **Solution.**

$$\vec{E} = E_x \hat{i} + E_y \hat{j} = E_{ox} \hat{i} \cos(-\omega t) + E_{oy} \hat{j} \cos(-\omega t + \frac{\pi}{2})$$

$$\vec{E} = E_{ox} \hat{i} \cos(-\omega t) + E_{oy} \hat{j} \sin(\omega t)$$

$$\vec{E} = E_{ox} \hat{i} \cos(\omega t) + E_{oy} \hat{j} \sin(\omega t)$$

$$\vec{E} = E_x \hat{i} + E_y \hat{j} = E_{ox} \hat{i} \cos(\theta) + E_{oy} \hat{j} \sin(\theta)$$

$$E_x = E_{ox} \cos\theta \quad \text{and} \quad E_y = E_{yx} \sin\theta$$

$$\frac{E_x}{E_{ox}} = \cos\theta \qquad \frac{E_y}{E_{oy}} = \sin\theta$$

$$\frac{E_x^2}{E_{ox}^2} + \frac{E_y^2}{E_{oy}^2} = \cos^2\theta + \sin^2\theta = 1$$

$$\boxed{\frac{E_x^2}{E_{ox}^2} + \frac{E_y^2}{E_{oy}^2} = 1}$$ is an equation of an ellipse $$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

Eccentricity is $$\boxed{\varepsilon = \sqrt{1 - \frac{b^2}{a^2}}}$$ => $$\boxed{\varepsilon = \sqrt{1 - \frac{E_{oy}^2}{E_{ox}^2}}}$$

HW O2. Quarter Wave Plate Design. A birefringent material allows for double refraction affecting perpendicular polarization states differently. The different wave speeds for each polarization state can allow us to retard one polarization state by $\pi/2$ and produce elliptically polarized light. Consider normal incidence and quartz, where for 590 nm light the index of refraction $n_e = 1.553$ for the extraordinary wave and the index of refraction for the ordinary wave $n_o = 1.544$. By the way, calcite is too brittle for a wave plate.

(a) Show that the relative phase shift between the light for the 2 polarization directions is

$$\Delta\phi = (2\pi / \lambda_0)\, d\, (n_e - n_o),$$

where λ_0 is the wavelength of the incoming light in vacuum or air, d is the thickness of the birefringent plate, and n_o, n_e are the indexes of refraction for the ordinary and extraordinary rays. If $d (n_e - n_o) = \lambda / 4$, then $\Delta\phi = \pi / 2$. But $d (n_e - n_o) = (4m + 1) \lambda / 4$ also works where m = 0, 1, 2, 3 …, i.e. an integral number of wavelengths plus $\lambda / 4$.

(b) Find the minimum thickness d, i.e. m = 0, tailored to 590 nm for quartz.

(c) Give the thickness for a quarter-wave quartz plate tailored to 590 nm where m = 200.

Solution (a)

Take: $\vec{E}_o = E_o \cos(k_o z - \omega t)\hat{k}$ and $\vec{E}_e = E_e \cos(k_e z - \omega t)\hat{k}$

since with refraction, the speed and wavelength change but not the frequency.

The relative phase shift as the waves travel a distance d is

$$\Delta\phi = k_e d - k_o d$$

$k(x - vt) = v\omega t$ => $kv = \omega$ or you might prefer to start with $v = \dfrac{\omega}{k}$

$k = \dfrac{\omega}{v}$ Index of refraction formula is $n = \dfrac{c}{v}$.

Combine these and you get $k = \dfrac{\omega}{v} = \omega \dfrac{n}{c} = 2\pi f \dfrac{n}{c} = 2\pi \dfrac{f}{c} n$.

In vacuum $c = \lambda_0 f$, where the problem stated the vacuum wavelengt is λ_0.

Note that $\omega = 2\pi f$ is the same everywhere for refraction problems.

Using $c = \lambda_0 f$ in the form of $\frac{f}{c} = \frac{1}{\lambda_0}$ in $k = 2\pi \frac{f}{c} n$

$$k = \frac{2\pi}{\lambda_0} n$$

Therefore $\Delta\phi = k_e d - k_o d$ becomes

$$\Delta\phi = \frac{2\pi}{\lambda_0} n_e d - \frac{2\pi}{\lambda_0} n_o d$$

$$\boxed{\Delta\phi = \frac{2\pi}{\lambda_0}(n_e - n_o)d}$$

Solution (b) To get circularly polarized light, the problem states from class that we need

$$\Delta\phi = \frac{2\pi}{\lambda_0}(n_e - n_o)d = \frac{\pi}{2}\text{, i.e., } (n_e - n_o)d = \frac{\lambda_0}{4}.$$

The problem also states that this quarter-wave can be realized in general for

$$(n_e - n_o)d = (4m+1)\frac{\lambda_0}{4}\text{, where } m = 0, 1, 2, 3, \ldots$$

For the thinnest quarter-wave plate, we want m = 0: $(n_e - n_o)d = \frac{\lambda_0}{4}$.

The values given are $n_e = 1.553$, $n_o = 1.544$, and $\lambda_0 = 590 \text{ nm}$.

$$d = \frac{\lambda_0}{4(n_e - n_o)} = \frac{590 \times 10^{-9} \text{ m}}{4(1.553 - 1.544)} = 1.64 \times 10^{-5} \text{ m} = 16.4 \text{ microns}$$

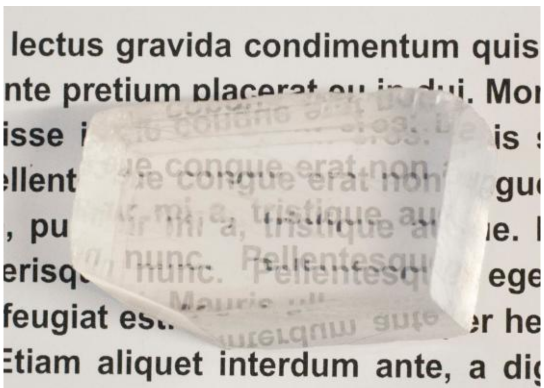
$$\boxed{16.4 \text{ microns}}$$

Solution (c) Case where m = 200.

$$(n_e - n_o)d = (4m+1)\frac{\lambda_0}{4} \quad => \quad (n_e - n_o)d = (4\cdot 200+1)\frac{\lambda_0}{4}$$

$$d = \frac{801\lambda_0}{4(n_e - n_o)} = 801\cdot(1.64\times 10^{-5})\ \text{m} = 0.0131\ \text{microns} = 13.1\ \text{mm}$$

$$\boxed{13.1\ \text{mm}}$$



HW O3. Birefringence in Rocks.

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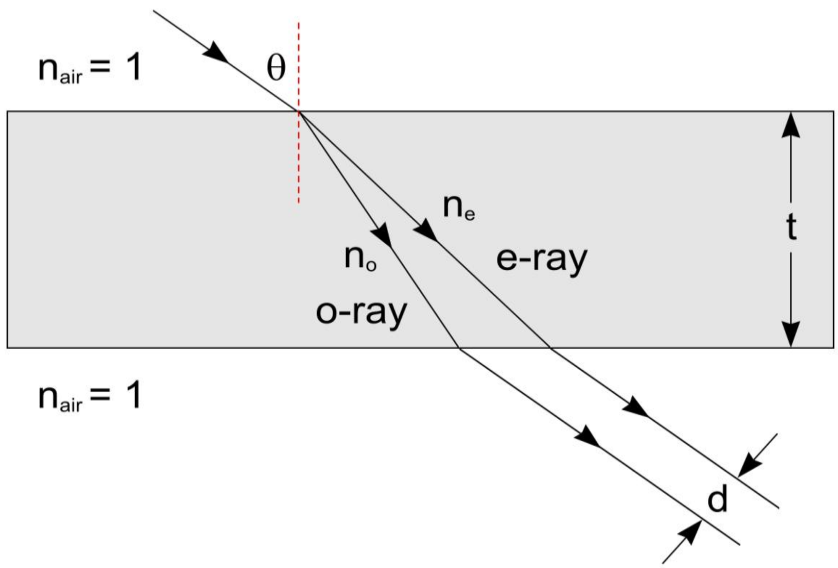
Near the wavelength of the sodium doublet, i.e., around 590 nm, the index of refraction for the ordinary ray and extraordinary wave are

$n_o = 1.658$ and $n_e = 1.486$.

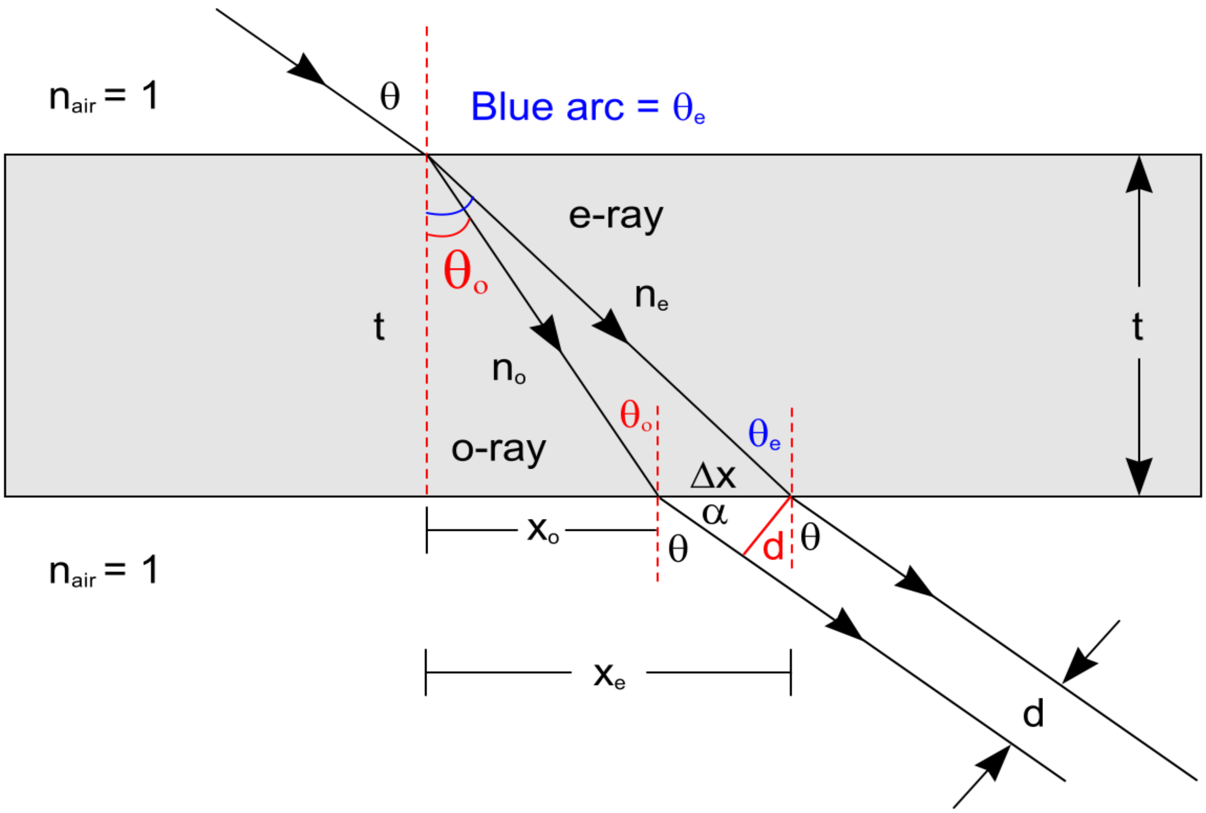
Pick up the calcite so that there is a layer of air between the print and the calcite.



Find d in terms of n_o, n_e, the incident angle θ, and thickness t.

Find d to 3 sig figs for calcite where $\theta = 30.0°$, and $t = 7.00\ \text{cm}$.

Solution (c)



$$\Delta x = x_e - x_o \qquad \tan\theta_o = \frac{x_o}{t} \qquad \tan\theta_e = \frac{x_e}{t}$$

$$\Delta x = t\tan\theta_e - t\tan\theta_o \quad => \quad \Delta x = t(\tan\theta_e - \tan\theta_o)$$

Note that $d = \Delta x \sin\alpha$ and $\alpha = \frac{\pi}{2} - \theta$. Then $d = \Delta x \cos\theta$.

Use $\Delta x = t(\tan\theta_e - \tan\theta_o)$ in $d = \Delta x \cos\theta$ to arrive at

$$d = t(\tan\theta_e - \tan\theta_o)\cos\theta .$$

$$d = t\left(\frac{\sin\theta_e}{\cos\theta_e} - \frac{\sin\theta_o}{\cos\theta_o}\right)\cos\theta$$

$$d = t\left(\frac{\sin\theta_e}{\sqrt{1-\sin^2\theta_e}} - \frac{\sin\theta_o}{\sqrt{1-\sin^2\theta_o}}\right)\cos\theta$$

Snell's Law gives $\sin\theta = n_o \sin\theta_o = n_e \sin\theta_e$

Substitute $\sin\theta_o = \dfrac{\sin\theta}{n_o}$ & $\sin\theta_e = \dfrac{\sin\theta}{n_e}$.

$$d = t\left(\frac{\sin\theta / n_e}{\sqrt{1-\sin^2\theta / n_e^2}} - \frac{\sin\theta / n_o}{\sqrt{1-\sin^2\theta / n_o^2}}\right)\cos\theta$$

$$d = t\left(\frac{1}{n_e}\frac{1}{\sqrt{1-\sin^2\theta / n_e^2}} - \frac{1}{n_o}\frac{1}{\sqrt{1-\sin^2\theta / n_o^2}}\right)\cos\theta\sin\theta$$

Given: $n_o = 1.658$, $n_e = 1.486$, $\theta = 30.0^\circ$, and $t = 7.00 \text{ cm}$.

$$\frac{1}{n_e}\frac{1}{\sqrt{1-\sin^2\theta / n_e^2}} = \frac{1}{1.486}\frac{1}{\sqrt{1-\sin^2 30^\circ / 1.486^2}} = 0.7146$$

$$\frac{1}{n_o}\frac{1}{\sqrt{1-\sin^2\theta / n_o^2}} = \frac{1}{1.658}\frac{1}{\sqrt{1-\sin^2 30^\circ / 1.658^2}} = 0.6326$$

$$d = 7 \text{ cm} \cdot (0.7146 - 0.6326)\cos 30^\circ \sin 30^\circ$$

$$d = 0.2485 \text{ cm}$$

$$\boxed{d = 2.49 \text{ mm}}$$