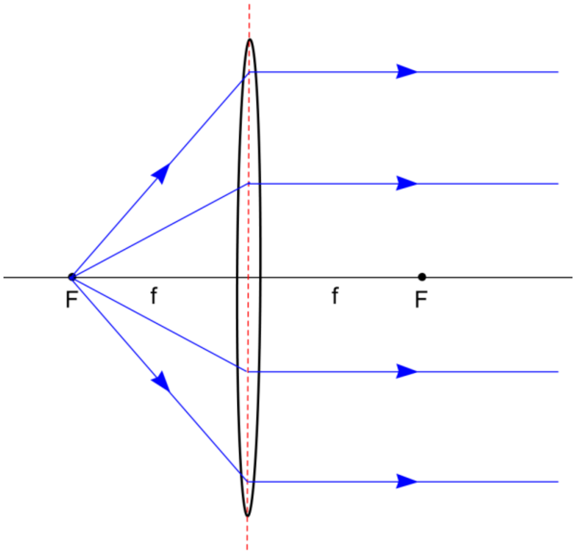
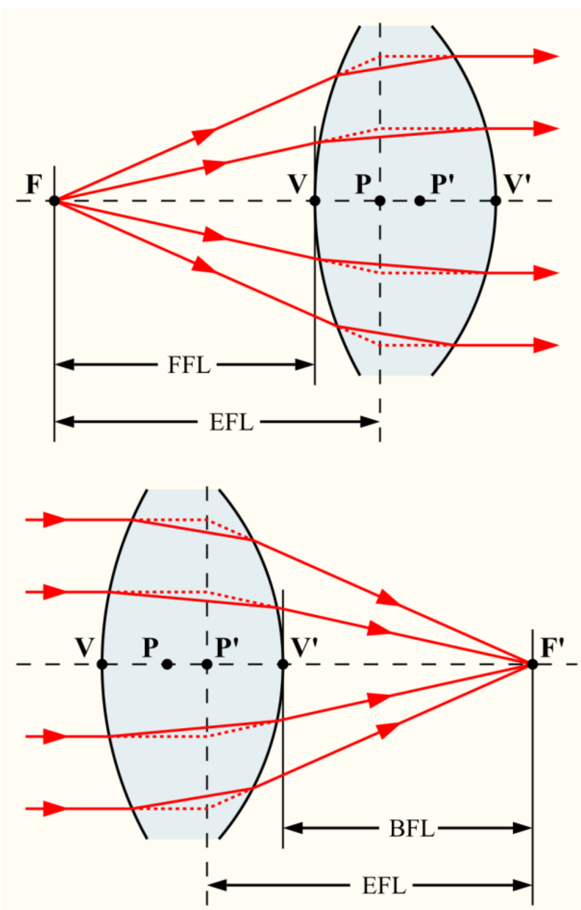
F1. Cardinal Points.



For a thin lens you have one focal length and it is the same on both sides. Light starting out at a focal point F results in parallel light.

The lens is so thin that we can replace it with the red dotted vertical line, actually a plane. This plane is called the principal plane.

In a thick lens there are two of these since the front and back focal lengths are not the same length. See the diagram below, where there is a front focal length FFL and a different back focal length BFL. These lengths will be different in the general case.



Wikipedia: DrBob AKA Bob Mellish

At the left is a thick converging lens. Note the two refractions for the top red ray. The ray appears to bend once at the primary principal plane located at point P on the optic axis.

But rays entering from the rear appear to bend once at the secondary principal plane that passes through point P'. In the thin lens above, these planes coincide and P is the same as P'.

The six labeled points are called cardinal points and they are described below.

F and F' are the front and rear focal points.
P and P' are the front and rear principal points.
V and V' are the front and rear surface vertices.

The three focal lengths are as follows:

FFL is the front focal length (from F to V).
BFL is the back focal length (from V' to F').
EFL is the effective focal length (from F to P).
We will show that EFL is also the distance from F' to P' as the derived formula will be symmetric in the lens parameters.

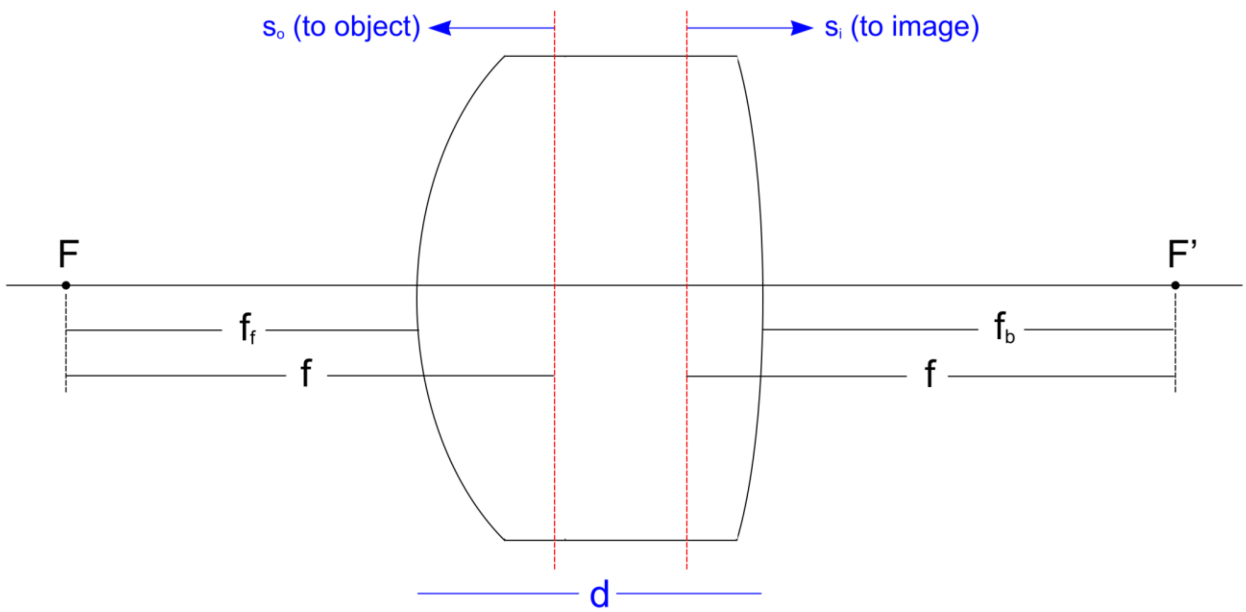
These three focal lengths are listed in the figure as

f_f for the front focal length, f_b for the front back length, f for the effective focal length.

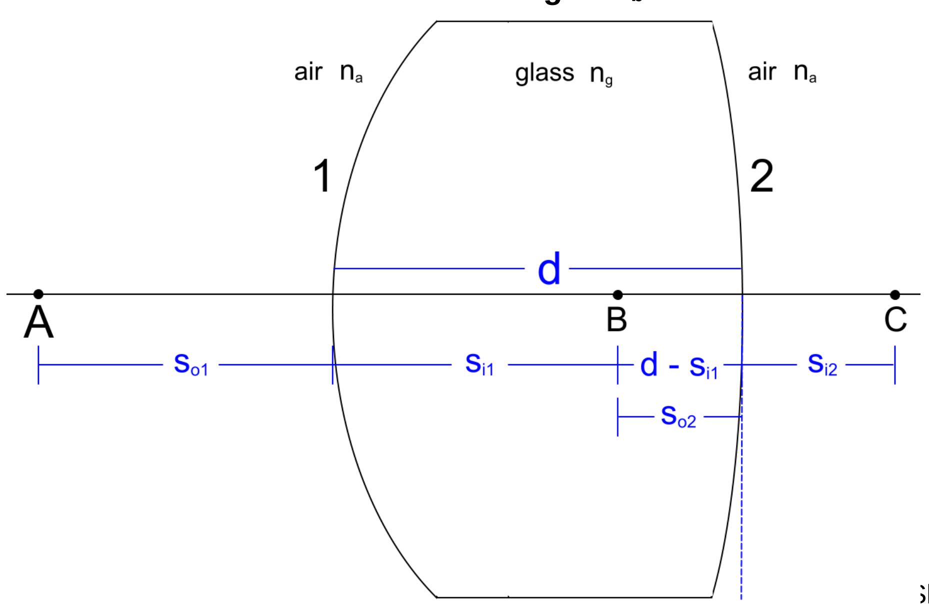
The effective focal length is the one satisfying

$$\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f},$$

all measured from the principal planes.



F2. Formula for the Back Focal Lengths f_b.



We start with two equations from the last chapter:

$$\frac{n_a}{s_{o1}} + \frac{n_g}{s_{i1}} = \frac{(n_g - n_a)}{R_1}$$

$$\frac{n_g}{s_{o2}} + \frac{n_a}{s_{i2}} = \frac{(n_a - n_g)}{R_2}$$

where $s_{o2} = d - s_{i1}$.

Setting $n_a = 1$ and $n_g = n$,

$$\frac{1}{s_{o1}} + \frac{n}{s_{i1}} = \frac{(n-1)}{R_1} \quad \text{and} \quad \frac{n}{s_{o2}} + \frac{1}{s_{i2}} = \frac{(1-n)}{R_2}.$$

The image due to surface 1 is in the glass in our figure, at point B. But it could be to the right of surface 2. Things would still work out. In that case $s_{o2} = d - s_{i1} < 0$. These formulas that we derived in the last class look like the thin-lens formula

$$\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f},$$

where for the first surface (surface 1):

$$s_0 = s_{o1}, \; s_i = \frac{s_{i1}}{n}, \text{ and } f_1 = \frac{R_1}{n-1}.$$

The magnification due to surface 1 is $M_1 = -\frac{s_i}{s_o} = -\frac{s_{i1}}{n s_{o1}}$.

For surface 2,

$$s_0 = \frac{s_{o2}}{n}, \; s_i = s_{i2}, \; f_2 = -\frac{R_2}{n-1}, \text{ and } M_2 = -\frac{s_i}{s_o} = -\frac{n s_{i2}}{s_{o2}}.$$

We eventually want an equation for s_{i2} without the intermediate image information s_{i1}. Let's work both equations in parallel, i.e., side by side and solve for the image distances in each case.

$$\frac{1}{s_{o1}} + \frac{n}{s_{i1}} = \frac{1}{f_1} \qquad \frac{n}{s_{o2}} + \frac{1}{s_{i2}} = \frac{1}{f_2}$$

$$\frac{n}{s_{i1}} = \frac{1}{f_1} - \frac{1}{s_{o1}} \qquad \frac{1}{s_{i2}} = \frac{1}{f_2} - \frac{n}{s_{o2}}$$

$$\frac{n}{s_{i1}} = \frac{s_{o1} - f_1}{f_1 s_{o1}} \qquad \frac{1}{s_{i2}} = \frac{s_{o2} - n f_2}{f_2 s_{o2}}$$

$$\frac{1}{s_{i1}} = \frac{s_{o1} - f_1}{n f_1 s_{o1}} \qquad \frac{1}{s_{i2}} = \frac{s_{o2} - n f_2}{f_2 s_{o2}}$$

$$s_{i1} = \frac{n f_1 s_{o1}}{s_{o1} - f_1} \qquad s_{i2} = \frac{f_2 s_{o2}}{s_{o2} - n f_2}$$

Now we use the connecting relation $s_{o2} = d - s_{i1}$.

$$s_{i2} = \frac{f_2 (d - s_{i1})}{(d - s_{i1}) - n f_2}$$

$$s_{i2} = \frac{f_2 d - f_2 s_{i1}}{d - s_{i1} - n f_2}$$

$$s_{i2} = \frac{f_2 d - f_2 s_{i1}}{d - n f_2 - s_{i1}}$$

Now look above and find where we found $s_{i1} = \dfrac{n f_1 s_{o1}}{s_{o1} - f_1}$.

We substitute it in the $s_{i2} = \dfrac{f_2 d - f_2 s_{i1}}{d - n f_2 - s_{i1}}$ equation.

And we start to get into some heavy algebra – and interesting.

If you are getting excited like I am, you are a theoretical physicist at heart.

$$s_{i2} = \frac{f_2 d - \dfrac{n f_1 f_2 s_{o1}}{s_{o1} - f_1}}{d - n f_2 - \dfrac{n f_1 s_{o1}}{s_{o1} - f_1}}$$

But to find the back focal length f_b we want

$$s_{o1} \to \infty \,.$$

We could have started with this limit first, but it is nice to see the general formula for s_{i2} and we need the general case in the next section to arrive at the front focal length.

When $s_{o1} \to \infty$, then $s_{i2} \to f_b$

and

$$s_{i2} = \frac{f_2 d - \dfrac{n f_1 f_2 s_{o1}}{s_{o1} - f_1}}{d - n f_2 - \dfrac{n f_1 s_{o1}}{s_{o1} - f_1}} \to \frac{f_2 d - n f_1 f_2}{d - n f_2 - n f_1} \text{ since } \frac{s_{o1}}{s_{o1} - f_1} \to 1 \,.$$

Math professors would like me to write for the last step $\dfrac{s_{o1}}{s_{o1} - f_1} \to 1$:

$$\lim_{x \to \infty} \frac{s_{o1}}{s_{o1} - f_1} = \lim_{x \to \infty} \frac{1}{1 - \dfrac{f_1}{s_{o1}}} = \frac{1}{1 - 0} = 1 \,.$$

So we have our back focal length.

$$f_b = \frac{f_2 d - n f_1 f_2}{d - n f_2 - n f_1}$$

$$\boxed{f_b = \frac{f_2 (d - n f_1)}{d - n(f_1 + f_2)}}$$

Did we make any mistakes? One way to acquire security in our answer, besides checking all the steps, is to see if in a limiting case that we get a result we already know. Suppose we have a thin lens. Then $d \to 0$ and

$$f_b = \frac{f_2 (d - n f_1)}{d - n(f_1 + f_2)} \to \frac{f_2 (0 - n f_1)}{0 - n(f_1 + f_2)} = \frac{n f_1 f_2}{n(f_1 + f_2)} = \frac{f_1 f_2}{(f_1 + f_2)} \text{, i.e.,}$$

$$\frac{1}{f_b} = \frac{1}{f_1} + \frac{1}{f_2} \text{, the thin lensmaker's formula result! What?}$$

Yes. From the last chapter

$$\frac{1}{f}=(n-1)\left[\frac{1}{R_1}-\frac{1}{R_2}\right]$$, which is equivalent to

$$\frac{1}{f_b}=\frac{1}{f_1}+\frac{1}{f_2},$$

since $f_1=\dfrac{R_1}{n-1}$ and $f_2=-\dfrac{R_2}{n-1}$.

F3. Formula for the Front Focal Lengths f_f.

The front focal length is found by letting $s_{i2}\to\infty$. Then $s_{o1}\to f_f$. Now we need that general formula we derived earlier.

$$s_{i2}=\frac{f_2d-\dfrac{nf_1f_2s_{o1}}{s_{o1}-f_1}}{d-nf_2-\dfrac{nf_1s_{o1}}{s_{o1}-f_1}}$$

We find $s_{i2}\to\infty$ when the denominator is set to zero.

$$d-nf_2-\frac{nf_1s_{o1}}{s_{o1}-f_1}=0$$

$$\frac{nf_1s_{o1}}{s_{o1}-f_1}=d-nf_2$$

Multiply both sides by $s_{o1}-f_1$.

$$nf_1s_{o1}=(s_{o1}-f_1)(d-nf_2)$$

$$nf_1s_{o1}=s_{o1}d-s_{o1}nf_2-f_1d+nf_1f_2$$

Get all the s_{o1} terms on the left side of the equation.

$$nf_1 s_{o1} + s_{o1} nf_2 - s_{o1} d = -f_1 d + nf_1 f_2$$

$$s_{o1}(nf_1 + nf_2 - d) = f_1\left(-d + nf_2\right)$$

$$s_{o1} = \frac{f_1\left(-d + nf_2\right)}{(nf_1 + nf_2 - d)}$$

$$s_{o1} = \frac{f_1\left(-d + nf_2\right)}{n(f_1 + f_2) - d}$$

$$s_{o1} = \frac{f_1\left(d - nf_2\right)}{d - n(f_1 + f_2)}$$

And this is our front focal length.

$$\boxed{f_f = \frac{f_1\left(d - nf_2\right)}{d - n(f_1 + f_2)}}$$

Let's see what happens for a thin lens. As $d \to 0$,

$$f_f = \frac{f_1\left(d - nf_2\right)}{d - n(f_1 + f_2)} \to \frac{f_1\left(0 - nf_2\right)}{0 - n(f_1 + f_2)} = \frac{nf_1 f_2}{n(f_1 + f_2)} = \frac{f_1 f_2}{(f_1 + f_2)}$$ and

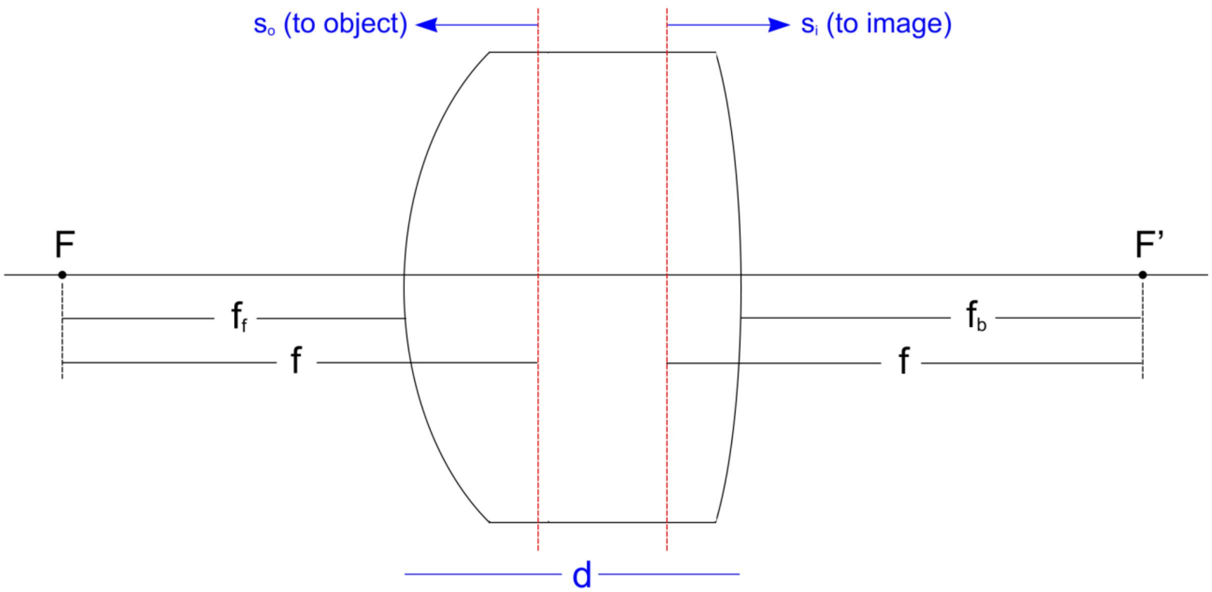
$$\frac{1}{f_f} = (n-1)\left[\frac{1}{R_1} - \frac{1}{R_2}\right]$$, the same result as for the back focal length.

For a thin lens, there is only one focal length

$$\frac{1}{f} = \frac{1}{f_b} = \frac{1}{f_f} = (n-1)\left[\frac{1}{R_1} - \frac{1}{R_2}\right].$$

F4. Gullstrand's Equation.

Our earlier figure is reproduced below.



Here are all our basic formulas reproduced.

$$\frac{1}{s_o}+\frac{1}{s_i}=\frac{1}{f} \qquad M=-\frac{s_i}{s_o}$$

$$\frac{1}{s_{o1}}+\frac{n}{s_{i1}}=\frac{1}{f_1} \qquad \frac{n}{s_{o2}}+\frac{1}{s_{i2}}=\frac{1}{f_2}$$

$$M_1=-\frac{s_{i1}}{ns_{o1}} \qquad M_2=-\frac{ns_{i2}}{s_{o2}}$$

$$s_{i1}=\frac{nf_1s_{o1}}{s_{o1}-f_1} \qquad s_{i2}=\frac{f_2s_{o2}}{s_{o2}-nf_2}$$

$$M=M_1M_2=\left[-\frac{s_{i1}}{ns_{o1}}\right]\left[-\frac{ns_{i2}}{s_{o2}}\right]=\frac{s_{i1}}{s_{o1}}\frac{s_{i2}}{s_{o2}} \text{ must match } M=-\frac{s_i}{s_o}$$

To find the effective focal length take $s_{o1} \to \infty$ in $\dfrac{1}{s_{o1}} + \dfrac{n}{s_{i1}} = \dfrac{1}{f_1}$. Then $s_{i1} \to nf_1$.

$$s_{o2} = d - s_{i1} \to d - nf_1$$

From $s_{i2} = \dfrac{f_2 s_{o2}}{s_{o2} - nf_2}$

$$s_{i2} = \frac{f_2 s_{o2}}{s_{o2} - nf_2} \to \frac{f_2(d - nf_1)}{(d - nf_1) - nf_2} = \frac{f_2(d - nf_1)}{d - n(f_1 + f_2)}$$

To keep our bearing, we summarize the parameters. For large s_o, $s_{o1} = s_o$.

$$s_{o1} = s_o \qquad s_{i1} = nf_1 \qquad s_{o2} = d - nf_1 \qquad s_i = f$$

Then $M = \dfrac{s_{i1}}{s_{o1}} \dfrac{s_{i2}}{s_{o2}} = -\dfrac{s_i}{s_o}$ leads to $M = \dfrac{nf_1}{s_o} \dfrac{s_{i2}}{d - nf_1} = -\dfrac{f}{s_o}$ giving

$$nf_1 \frac{s_{i2}}{d - nf_1} = -f \qquad s_{i2} = -\frac{f(d - nf_1)}{nf_1}.$$

We need our former formula $s_{i2} = \dfrac{f_2(d - nf_1)}{d - n(f_1 + f_2)}$ and use it with $s_{i2} = -\dfrac{f(d - nf_1)}{nf_1}$.

$$\frac{f_2(d - nf_1)}{d - n(f_1 + f_2)} = -\frac{f(d - nf_1)}{nf_1}$$

$$\frac{f_2(d - nf_1)nf_1}{d - n(f_1 + f_2)} = -f(d - nf_1)$$

The $(d - nf_1)$ factors cancel.

$$\frac{f_2 n f_1}{d - n(f_1 + f_2)} = -f$$

$$f = -\frac{n f_1 f_2}{d - n(f_1 + f_2)}$$

$$f = \frac{n f_1 f_2}{n(f_1 + f_2) - d}$$

$$\frac{1}{f} = \frac{n(f_1 + f_2) - d}{n f_1 f_2}$$

$$\frac{1}{f} = \frac{n(f_1 + f_2)}{n f_1 f_2} - \frac{d}{n f_1 f_2}$$

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{n f_1 f_2}$$

This formula is Gullstrand's Equation.

Note that in the limit of a thin lens, $d \to 0$ and we recover the expected result from before.

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}$$

F5. Lensmaker's Formula for the Thick Lens.

We now remember that we derived $f_1 = \frac{R_1}{n-1}$ and $f_2 = -\frac{R_2}{n-1}$, which we can write as

$$\frac{1}{f_1} = \frac{n-1}{R_1} \quad \text{and} \quad \frac{1}{f_2} = -\frac{n-1}{R_2}.$$

Then Gullstrand's equation $\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{nf_1 f_2}$ becomes

$$\frac{1}{f} = \frac{n-1}{R_1} - \frac{n-1}{R_2} - \frac{d}{n}\left[\frac{n-1}{R_1}\right]\left[-\frac{n-1}{R_2}\right]$$

$$\frac{1}{f} = \frac{n-1}{R_1} - \frac{n-1}{R_2} + \frac{d}{n}\left[\frac{n-1}{R_1}\right]\left[\frac{n-1}{R_2}\right]$$

$$\frac{1}{f} = (n-1)\left[\frac{1}{R_1} - \frac{1}{R_2} + \frac{(n-1)d}{nR_1R_2}\right]$$

Lensmaker's formula for thick lens!

With $d \to 0$ we recover the Lensmaker's formula for a thin lens.

The Lensmaker Shows Up Again