Normal Gravity. Prof. Ruiz (Doc), UNC-Asheville (1978-2021), DoctorPhys on YouTube.
Chapter A. Syllabus. Prerequisites: Physics I, II. Calculus I, II, III.

A1. Course Description. These notes and accompanying YouTube videos at DoctorPhys are developed for ambitious students with at least one year of calculus-based physics and three semesters of calculus. The goal is to cover all the mathematical physics necessary to show all steps in deriving Somigliana's formula for normal gravity. The course consists of Chapters labeled A to Z, as listed in the table below. During the journey we take many detours so you can develop your mathematical physics skills and master material related to normal gravity.

A. Syllabus (This Chapter)
B. Algebra and Trig (Review)
C. Calculus (Review)
D. Physics (Review)
E. Infinite Series
F. Taylor and Fourier Series
G. Gravitation
H. Conic Sections
I. PI, Areas of Circles and Ellipses
J. Ellipse Perimeter
K. Ellipsoids
L. Cartesian Coordinates
M. Polar Coordinates
N. Elliptic Coordinates I
O. Elliptic Coordinates II
P. Curvilinear Coordinates
Q. Cylindrical Coordinates
R. Spherical Coordinates
S. Laplacian Applications
T. Ellipsoidal Coordinates
U. World Geodetic System 1984 (WGS 84)
V. Geodetic Coordinates
W. The Geopotential
X. Legendre Polynomials
Y. Legendre Functions (1st, 2nd Kind)
Z. Normal Gravity

The Somigliana formula gives *normal* gravity at all surface points when the Earth is modeled as a rotating uniform ellipse of revolution. *Normal* means that the gravity force vector is the force component perpendicular to the surface of the Earth at the desired position. The rotation of the Earth is important in determining the effective gravity. The net gravity force vector does not point to the exact center of the Earth as it would for a non-rotating sphere. But normal gravity is the perpendicular force component that gives what a scale would read for your weight.

A2. Somigliana's Formula for Normal Gravity.

Carlo Somigliana (1860-1955)
Source: Wikipedia (Public Domain)

An exact formula for normal gravity for a rotating ellipsoid was first derived by Carlo Somigliana, who was an Italian mathematical physicist.

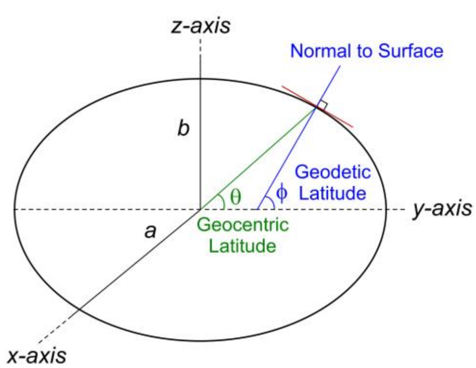
He derived his gravity formula in 1929. When one uses an ideal ellipsoidal model to calculate gravity, we often we refer to gravity as theoretical gravity.

The Somigliana formula is

$$\gamma = \frac{a\gamma_e \cos^2\phi + b\gamma_p \sin^2\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}}\text{, where}$$

γ is gravity (m/s^2), γ_e is gravity at the equator, γ_p is gravity at the poles,
ϕ is the angle where the normal from the surface meets the equatorial plane (see figure),
a is the semi-major axis (equatorial radius), and b is the semi-minor axis (polar radius).

I will show you later in detail how to transform among the various angles. The form of Somigliana's formula is very nice for the following reasons.



1) It is fairly easy to remember since the semi-major axis a goes with g_e (gravity at the equator) and the semi-minor axis b is paired with g_p (gravity at the poles).

2) The geodetic angle ϕ is the latitude of geography used for our locations on Earth.

3) The geodetic angle ϕ is the one chosen by the *World Geodetic System 1984* for theoretical gravity formulas.

Note that the above ellipsoid is highly exaggerated. The actual shape of the Earth is very close to a sphere. We use γ for theoretical gravity, where we consider the Earth as an ideal uniform oblate spheroid. The notation g is reserved for the actual acceleration due to gravity, which takes into consideration the nonuniform density of the Earth. In Somigliana's original formula, he used g back in 1929. An average g often used in physics is $g = 9.8\ \frac{\text{m}}{\text{s}^2}$. Some like to let $\gamma_e = \gamma_a$ and $\gamma_p = \gamma_b$.

A3. Ladislav Hora. Much of my inspiration for this course comes from the following publication.

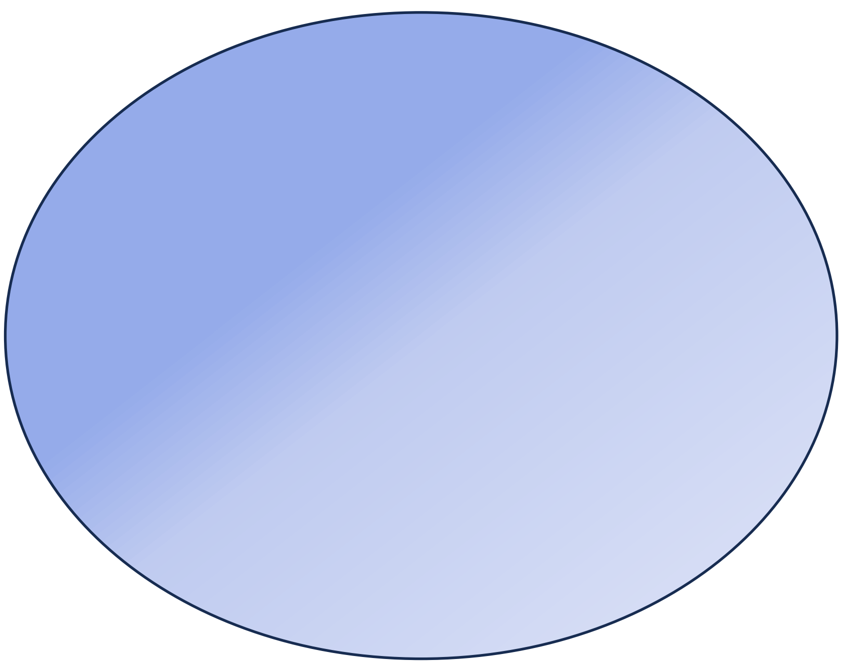
> Hora, Ladislav (1971). "On the Somigliana's formula," *Aplikace matematiky*, **16**, 98-108. Link to pdf at the Czech Digital Mathematics Library. Here is a link to their main page.

In this paper, Hora gives a most elegant derivation of the Somigliana formula. This derivation will be the climax of our course in the last chapter. His method is similar to that found in the following excellent publication that appeared about 10 years before his paper.

> Molodenskii, Mikhail S., Eremeev, Vladimir F., and Yurkina, Mariya I. (1962). *Methods for Study of the External Gravitational Field and Figure of the Earth*. Jerusalem: Israel Program for Scientific Translations.

The author Ladislav Hora (1932-2013) was born on August 9, 1932 and died on March 19, 2013. He was a geodesist, specializing in higher geodesy issues and field work. Source: Hora, Ladislav (1932-2013).

Hora was affiliated with the Czech Republic and inducted as a fellow of the International Association of Geodesy (IAG). See the Springer handbook Drewes, Hermann, Kuglitsch, Franz, Adám, József, and Rózsa, Szabolcs (2016). "The Geodesist's Handbook 2016," *Journal of Geodesy* **90**, 907–1205. Springer, on behalf of the International Association of Geodesy (IAG). From the directory of the registered participants in the XXIII International Union of Geodesy and Geophysics (IUGG) General Assembly in Sapporo in 2003, Hora's entry is "Hora, L., Dr., F, Czech. PolyTech. Univ., Takurova 7, 166 29 Praha 6-Dejvice. Czech Republic."



A4. Course Intent. My goal in many of my courses is to introduce students with a full year of calculus-based physics and three semesters of calculus to advanced material, applying mathematical physics. In this case, we are going to derive the Somigliana's formula for normal gravity, an important formula in the field of *geodesy*. The National Oceanic and Atmospheric Administration (NOAA) gives the following definition for *geodesy*.

> Geodesy is the science of accurately measuring and understanding the Earth's geometric shape, orientation in space, and gravity field. (NOAA)

Our journey starts off with a review of high school trigonometry. But we will be at the advanced undergraduate level and graduate level in many sections. Here is why. During the last semester of my four-year physics major, I took a one-semester quantum mechanics course. We solved the Schrödinger equation for the hydrogen atom, which included the Laplacian $\nabla^2 V$ in spherical coordinates. I will explain what the Laplacian is later. In our course, we solve the Laplacian in oblate ellipsoidal coordinates, which is definitely much harder than spherical coordinates.

You need to be aware that in geodesy, you are most likely to see Δ for the Laplacian instead of ∇^2. Though I usually try to stick with notation in the field I am discussing, I will use ∇^2 in this case, the preferred notation in theoretical physics. One reason that I am keeping with ∇^2 is that I use Δ for change very often in my courses, an example being $\Delta x = x_2 - x_1$. But, as a mathematics professor once told me, notation has a strong subjective component and that's why in many math books the notation for the text is given on the inner sides of the front and back covers for the book.

I have seen papers published in atmospheric science where a whole page is devoted to listing all the symbols used in the paper and what they stand for. It would be nice if everyone agreed on a standard notation for everything, but that ideal is not reality. Therefore, be prepared when you consult other sources. You will find the definitions of angles such as θ and ϕ meaning the opposite of each other, even when two authors are discussing the same problem.

What I particularly like about the derivation of Somigliana's formula is that so many areas of mathematical physics are needed. In teaching, we often use textbooks and papers written many years after the initial discoveries. The reason is that decades after a discovery, the methods of derivation become much more accessible to students, both in notation and approach. I have consulted many books and references on our subject of geodesy. I have encountered, in important classical texts where ellipsoidal coordinates are introduced, figures representing three dimensions that I have found extremely difficult to grasp. My own approach, is inspired by the already mentioned excellent paper by Hora (1971) and text by Molodenskii, Eremeev, and Yurkina (1962). The traditional brain-twisting ellipsoidal diagrams are replaced in this approach with clear mathematical steps. I also have two full chapters on elliptic coordinates before we rotate the ellipse about the z-axis to delve into ellipsoidal coordinates in 3D.

My intent is to show all steps, even deriving the Laplacian in oblate ellipsoidal coordinates, as well as showing where the Legendre polynomials come from by solving Legendre's differential equation with a power series. Following through this course will enable you to use so many powerful tools common in mathematical physics, which I often also refer to as theoretical physics. As a physicist, I am relatively new to the field of geodesy and I was surprised that my background in mathematical physics prepared me well for entering this new field of study. I hope your study is as rewarding an experience as it was for me.

Since our course includes so much background material, it serves as a mathematical toolbox.

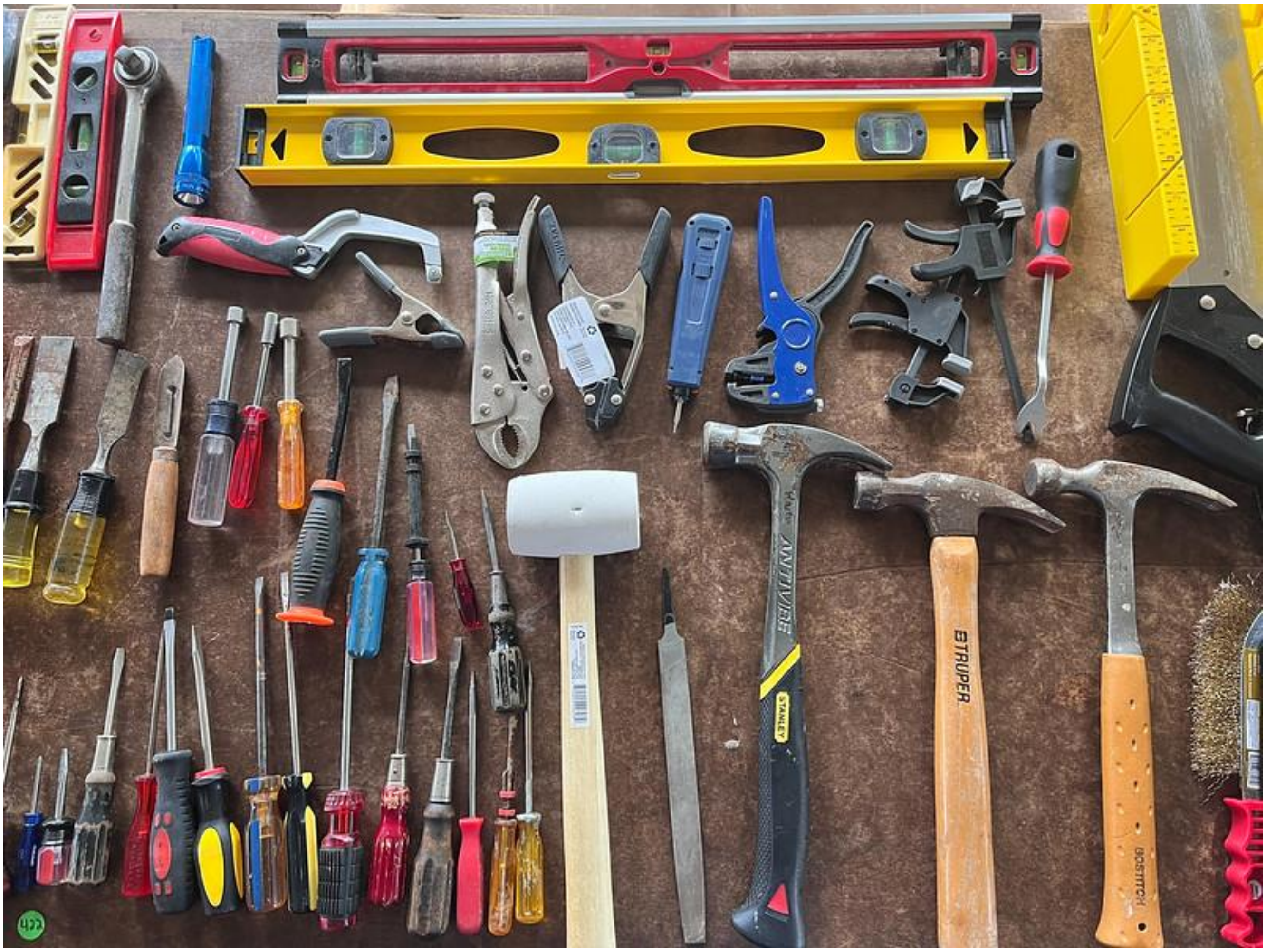


Table of Hand Tools. Courtesy Jonathan Cutrer, flickr.

A5. Geodesy and Related Fields. My interest in gravity on the Earth's surface and in the troposphere arose from my interest in studying temperature lapse rates in atmospheric science. Therefore, I think of these notes as a spin-off of my work in meteorology. However, I would like my treatment here to be self-contained with a unified aim: to derive the Somigliana formula including steps that are often left to the reader in textbooks. The Somigliana formula gives gravity at the Earth's surface anywhere on the ideal Earth model.

The subject of *geodesy* focuses on describing the shape of the Earth in terms of coordinate systems (geometric geodesy) and its gravitational field at different points on the Earth (physical geodesy). Geodesy is important for surveying and also for the global-positioning-satellite system.

Surveyors on the Author's Driveway

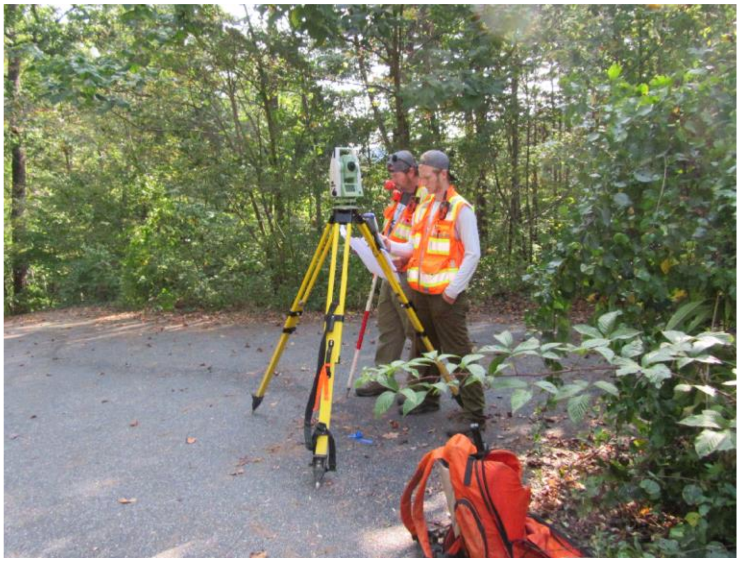
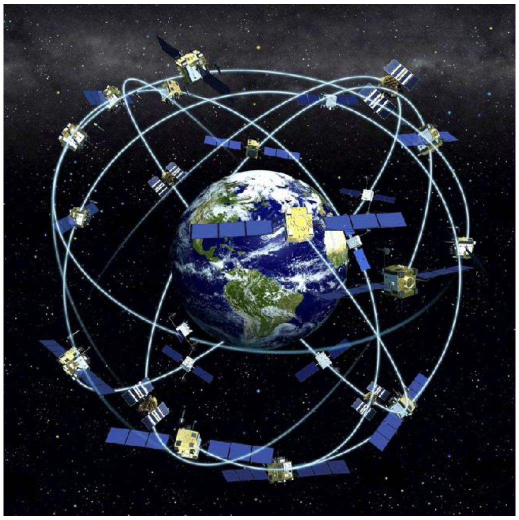


Photo by Author (September 29, 2017)

Global-Positioning Satellite (GPS) System



Credit NOAA

A6. Course Philosophy. The philosophical approach I am taking here is one where we derive everything from first principles. There are two main advantages for spending the time with such detail.

a. Profound Understanding. You acquire deep understanding when you see where everything comes from. You are empowered to do your own research.
b. Powerful Mathematical Techniques. You learn mathematical techniques that are common across the physical sciences and engineering.

I am amazed that all the fundamental mathematics and physics required to understand geodesy I have encountered in my studies of physics. This observation also extends to fields in engineering. Therefore, by deriving everything, we acquire a foundation that is applicable well beyond geodesy. An analogy here would be learning how to use generic tools in a toolbox. You use the same basic tools to fix anything. By working through a derivation, you are fixing something with your tools. When, you skip the derivations, it is like having someone else repair things for you.

When one takes a course in mathematical physics, one learns about these basic tools, but it can get tiring since there is no single overall specific application in mind. In contrast, our course has a specific goal, which then serves as an incentive. However, we do not cover all the tools found in a comprehensive mathematical methods course. We focus on the tools required for the task at hand – and these are many. It is still advisable to consider taking a general course on mathematical methods. Such courses go by several names, including mathematical physics, mathematical methods for engineers, theoretical physics, etc.

An important ingredient in the approach taken in such courses of applied mathematics, and the one we take here, is that we strike a balance between formal derivation and application. On average, our derivations are informal rather than rigorous mathematical proofs. However, some derivations are quite rigorous. In the next section I will address the difference with a specific example.

The Blue Marble

Courtesy NASA via Wikipedia

A7. Derivations. Derivations encountered in physics courses often do not please mathematicians. Before illustrating by example an informal proof and rigorous one, here a some general examples.

- The Dirac Delta Function. Theoretical physicist Paul Dirac (1902-1984) defined a function that is zero for all x and infinite at $x = 0$. When I first taught quantum mechanics and introduced Dirac's delta function, a double major in math and physics became quite upset. So, I researched the topic and then approached the math professor who was discussing this problem with my student behind my back. I showed the math professor a mathematical physics text that started with Gaussian functions, bell-shaped curves with the same area. When these became taller and narrower approaching what Dirac had in mind, the mathematician was happy. We could take derivatives of the Gaussians. Those functions were continuous. All was well. But physicists do not take the time to treat the Dirac delta function this way.
- Quantum Mechanics. When quantum mechanics was developed in the 1920s you might say physicist were cavalier in their use of math and infinite numbers of wavefunctions to describe general states of a bound particle. The great mathematician David Hilbert (1862-1943), working with physicists and other mathematicians, paved the way for making quantum mechanics stand on firm mathematical ground. The infinite-dimensional space in which the solutions can be expressed is called Hilbert space in his honor.
- Green's Functions in Electrodynamics. When I was an undergraduate, I was studying a graduate text in electrodynamics for some material related to my undergraduate research project. *Classical Electrodynamics* by John David Jackson is a great text that has been used in graduate school for decades. It is considered a classic and rigorous text (for physicists, as you will see below). One day, I went to a mathematician to ask for some insight that could possibly help me and my physics research advisor in our work. The math professor looked at the page I showed him. He started to look disgusted. Then he turned a few pages, shaking his head and stated: "This is not mathematics." So, you get my point. What is good enough for a physicist is not always good enough for a mathematician.

I would like to give a specific simple example of what I mean by pleasing a physicist and pleasing a mathematician. I will refer to what pleases the physicist as "the informal derivation" and what pleases the mathematician will be called "the rigorous mathematical proof." The example I present below is to find a formula for the sum of the first n positive odd integers.

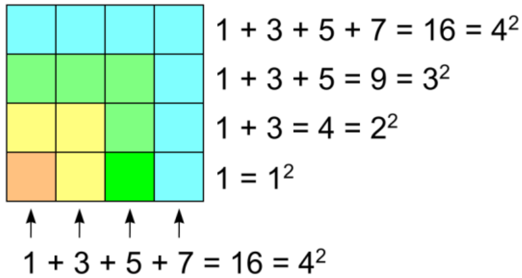
$$S_n = 1 + 3 + 5 + 7 \ldots 2n - 1 \qquad n = 1, 2, 3 \ldots$$

Before proceeding, we verify that $2n - 1$ is the n$^{\text{th}}$ positive odd integer for the first few cases.

n	1	2	3	4	5	6	7
$2n-1$	$2\cdot1-1=1$	$2\cdot2-1=3$	$2\cdot3-1=5$	$2\cdot4-1=7$	$2\cdot5-1=9$	$2\cdot6-1=11$	$2\cdot7-1=13$

We would like to derive the formula $S_n = 1 + 3 + 5 + 7 \ldots 2n - 1 = n^2$.

a. Informal Derivation. We derive our formula for $S_n = 1+3+5+7 \ldots 2n-1$ by constructing a box of numbers as shown below.



Focus on the four following areas:

i) the 1 x 1 square with 1 little square,
ii) the 2 x 2 square with 4 little squares,
iii) the 3 x 3 square with 9 little squares,
iv) the 4 x 4 square with 16 little squares.

Notice how each sum of odd integers matches the area of its corresponding square. We infer from our observations that the formula is

$$S_n = 1+3+5+7 \ldots 2n-1 = n^2 .$$

b. Formal Rigorous Proof. We will employ proof by induction. We assume the formula is true for the n^{th} case and then show it is true for the $(n + 1)^{th}$ case. We then show it is true for the first case, which completes the rigorous proof that the formula is true for all cases.

i) Assume

$$S_n = 1+3+5+7 \ldots 2n-1 = n^2 .$$

Then we proceed with $S_{n+1} = 1+3+5+7 \ldots 2n-1+2n+1$ and see if we get $(n+1)^2$.

$$S_{n+1} = 1+3+5+7 \ldots 2n-1+2n+1 \quad => \quad S_{n+1} = (1+3+5+7 \ldots 2n-1)+(2n+1)$$

Since we are assuming $S_n = 1+3+5+7 \ldots 2n-1 = n^2$, we have

$S_{n+1} = n^2 + (2n+1)$, which we can write as

$$S_{n+1} = (n+1)^2 .$$

ii) Demonstrate the Formula $S_n = 1+3+5+7 \ldots 2n-1 = n^2$ is true for the First Case.

For $n = 1$, the sum of the first odd integer is $S_1 = 1$.

The formula $S_n = n^2$ gives $S_1 = 1^2 = 1$.

Therefore, $S_n = 1+3+5+7 \ldots 2n-1 = n^2$ is true for all cases from 1 to infinity!

Though the rigorous proof is the gold standard for mathematicians, the informal derivation presented earlier is perfectly fine for an engineer or physicist.

However, many of our derivations will be very similar to those found in math books and will please even strict mathematicians. But in other instances, mathematicians will want more. We will not distinguish between the two in our derivations. I appreciate having taken three semesters of calculus with mathematicians in college and seeing rigorous proofs. Then, later, applying math in physics, I began to also appreciate the “informal” derivations of physicists. If we insisted on formal derivations for everything, we would not have time to do the physics. We would be exhausted.

Talking about exhaustion, I must warn you that theoretical physics often includes extremely lengthy calculations. If you like doing a 1000-piece jigsaw puzzle, you will probably look forward to calculations and the elation when you complete them. This course will give you great strength in mathematical methods. On the other hand, if you prefer to be given a formula and use it, then, this course is not for you. Since I have already given you Somigliana’s formula, we are finished. You have the formula. And there is nothing wrong with doing this. I acquired respect for looking up formulas when I was an Adjunct Professor of Mechanical Engineering at North Carolina State University in the 1980s. If you are doing engineering, you need to look up formulas and data in numerical tables to use them in your challenging engineering applications. There is no time to derive everything. So, it depends on your goals. If you want to become a grandmaster in understanding the fundamentals of much applied mathematics and physics, then this course is for you.

I often start my courses with some review material from introductory courses. When we review ellipses, we will actually be going back to high school algebra. In other cases, we will review material from introductory physics and calculus. From this foundation, we will proceed to material treated in upper-level science courses. Finally, we will reach material at the level of graduate school. I hope you enjoy the journey.

Michael Ruiz, Ph.D.
Asheville, North Carolina, USA
August 31, 2024

P.S. (December 23, 2024). Math Physics helped me psychologically deal with Hurricane Helene. Helene hit my hometown the morning of September 27, 2024. We heard trees falling throughout the night. Trees hit the house, my fence in many places, our backyard forest was wiped out, electricity went out, and water went out. Life changed for us drastically. Days were spent with neighbors clearing our own roads and driveways. But later we had to pay much to have professionals do tree removal. We often went with a neighbor who had a truck to get flushing water from a nearby stream, which later was run through a purification system for us as drinkable water. We car pooled in the early days to the one grocery store open to get food and ice to preserve the food for two days. Electricity returned in two weeks but tree removers tore up Internet lines. Water came back in weeks and eventually became drinkable. House repairs are still underway at this time (December 2024). I discovered that I could take great solace in working on this text with pen and paper during the daylight hours when taking breaks from my new daily chores. I learned that mathematical physics can be strong mental medicine. If you are interested in Helene videos, as of this writing, I have hours of video spanning weeks, which I plan to post at my personal YouTube channel: DrMichaelRuiz. If you are interested, see the Playlist labeled "Helene" at that YouTube channel.

Author's Backyard After Hurricane Helene (New Deck in Foreground Not Completed)

Photo by Author, November 23, 2024, Asheville, North Carolina, USA.

A8. Topics. Below is a detailed syllabus.

A. Syllabus
1. Course Description
2. Hora Ladislav
3. Carlo Somigliana
4. Course Intent
5. Geodesy and Related Fields
6. Course Philosophy

Bibliography

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