Class Q. Laplace Transforms

Q1. The Laplace Transform

Trick learned from Dr. Mattuck MIT

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$A(x) = \sum_{n=0}^{\infty} a_n x^n \Delta n \quad \Delta n = 1$$

i) $\Delta n \to dn$
ii) rip off subscripts $a_n \to a(n)$
iii) $\Sigma \to$ snake

$$A(x) = \int_0^{\infty} a(n) x^n \, dn$$

$x^n = [e^{\ln x}]^n$, $x = e^{\ln x}$, $\ln x = \ln(e^{\ln x})$

we want $\ln x = -s$, $s > 0$

Since n is usually used for integers, replace with t. Also, use $f(t)$.

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt$$

Laplace Transform

Q2. Evaluating Laplace Transforms

Remember $s > 0$

$f(t) = 1$: $F(s) = \int_0^{\infty} e^{-st} dt = \left. \dfrac{e^{-st}}{-s} \right|_0^{\infty} = 0 - \left(-\dfrac{1}{s}\right) = \dfrac{1}{s}$

$f(t) = e^{at}$: $F(s) = \int_0^{\infty} e^{at} e^{-st} dt$ We must have $s > a$

$$F(s) = \left. \frac{e^{(a-s)t}}{a-s} \right|_0^{\infty} = 0 - \frac{1}{a-s} = \frac{1}{s-a}$$

$g(t) = e^{at} f(t)$

$$G(s) = \int_0^{\infty} e^{at} f(t) e^{-st} dt = \int_0^{\infty} f(t) e^{-(s-a)t} dt$$

consider $s-a$ as a unit

Shifting Property $G(s) = F(s-a)$

$f(t)$: $\cos\omega t$, $\sin\omega t$ Real-Imaginary Trick

$e^{i\omega t} = \cos\omega t + i\sin\omega t$

$L\{e^{at}\} = \dfrac{1}{s-a}$ $a = i\omega$

$L\{e^{at}\} = \dfrac{1}{s-i\omega} \dfrac{s+i\omega}{s+i\omega} = \dfrac{s+i\omega}{s^2+\omega^2}$

$L\{\cos\omega t\} = \dfrac{s}{s^2+\omega^2}$ $L\{\sin\omega t\} = \dfrac{\omega}{s^2+\omega^2}$

$L\{e^{-at}\cos\omega t\} = \dfrac{s+a}{(s+a)^2+\omega^2}$ $L\{e^{-at}\sin\omega t\} = \dfrac{\omega}{(s+a)^2+\omega^2}$

shifting $s \to s+a$

t^n $L\{t^n\} = \int_0^{\infty} t^n e^{-st} dt = \dfrac{n!}{s^{n+1}}$

You used derivative trick + did this for Homework.

See text for the figures + Laplace Transform Table

Q3. Laplace Transform of Derivative

$$L\{f'(t)\} = \int_0^\infty \frac{df}{dt} e^{-st}\,dt$$

$$\frac{d[fe^{-st}]}{dt} = \frac{df}{dt}e^{-st} - fse^{-st}$$

always think of integration by parts as related to product rule of differentiation

$$L\{f'(t)\} = \int_0^\infty \frac{d}{dt}[fe^{-st}]\,dt + \int_0^\infty s f(t) \overset{e^{-st}}{\downarrow} dt$$

$$L\{f'\} = fe^{-st}\Big|_0^\infty + sL\{f(t)\}$$

independent of t (constant)

Important that $f/e^{st} < 1$

$$L\{f'\} = 0 - f(0) + sL\{f\}$$

$$\boxed{L\{f'(t)\} = sF(s) - f(0)}$$

$$L\{f''(t)\} = \underbrace{L\{g'(t)\}}_{sG(s) - g(0)} \qquad g(t) = f'(t)$$

$$\boxed{L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)} \quad \llcorner\, sF(s) - f(0)$$

Q4. Differential Equations

$$\frac{dN(t)}{dt} = -\lambda N(t)$$

Transforms diff. eq. into an algebraic one. We all prefer algebra.

1. Take Laplace Transform $\quad sF(s) - N(0) = -\lambda F(s)$
2. Solve the Algebraic Eq. $\quad F(s)(s+\lambda) = N(0)$

$$F(s) = \frac{N(0)}{s+\lambda}$$

3. Use Laplace Table to Transform Back

$$N(t) = L^{-1}\{F(s)\} = N(0)\,L^{-1}\overbrace{\left\{\frac{1}{s+\lambda}\right\}}^{e^{-\lambda t}}$$

$$N(t) = N(0)e^{-\lambda t} \qquad e^{at} \leftrightarrow \frac{1}{s-a}$$

BONUS Review

Q5. Damped Harmonic Oscillator

See text $\quad \llcorner$ from Intro Physics

You will do a homework problem along these lines.

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