Theoretical Physics
Prof. Ruiz, UNC Asheville
Chapter B Solutions. What is e? Euler's Formula, Integral Tricks

HW-B1. An Integral Family with Exponential Decay. Use the standard method to evaluate an integral similar to

$$\int_0^\infty e^{-x}\,dx$$ that you will need for the next part.

.

Then use one of our tricks to determine the general result in terms of n for

$$\int_0^\infty x^n e^{-x}\,dx\,,$$

where n = 0, 1, 2, ... What should you do if you do not see a parameter that you can use for differentiation?

Solution

$$I(a) = \int_0^\infty e^{-ax}\,dx = \left.\frac{e^{-ax}}{-a}\right|_0^\infty = 0 - \left[\frac{1}{-a}\right] = \frac{1}{a}$$

$$\int_0^\infty x e^{-ax}\,dx = \left[-\frac{d}{da}\right]\int_0^\infty e^{-ax}\,dx = \left[-\frac{d}{da}\right]\frac{1}{a} = \frac{1}{a^2}$$

$$\int_0^\infty x^2 e^{-ax}\,dx = \left[-\frac{d}{da}\right]\int_0^\infty x e^{-ax}\,dx = \left[-\frac{d}{da}\right]\frac{1}{a^2} = \frac{2}{a^3}$$

$$\int_0^\infty x^3 e^{-ax}\,dx = \left[-\frac{d}{da}\right]\int_0^\infty x^2 e^{-ax}\,dx = \left[-\frac{d}{da}\right]\frac{2}{a^3} = \frac{3\cdot 2}{a^4}$$

$$\int_0^\infty x^4 e^{-ax} dx = \left[-\frac{d}{da}\right] \int_0^\infty x^3 e^{-ax} dx = \left[-\frac{d}{da}\right] \frac{3\cdot 2}{a^4} = \frac{4\cdot 3\cdot 2}{a^5}$$

Psyche out the pattern.

$$\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$$

Now set $a = 1$.

$$\boxed{\int_0^\infty x^n e^{-x} dx = n!}$$

SHORT-CUT NOTATION METHOD.

$$\int_0^\infty x^n e^{-ax} dx = \left[-\frac{d}{da}\right]^n \int_0^\infty e^{-ax} dx = \left[-\frac{d}{da}\right]^n \frac{1}{a}$$

$$\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$$

$$\boxed{\int_0^\infty x^n e^{-x} dx = n!}$$

HW-B2. Some Integrals with Gaussians. Use a standard method to evaluate an integral similar to

$$\int_0^\infty x e^{-x^2}\, dx$$ that you will need for the next part.

Then use one of our tricks to determine the general result in terms of n for

$$\int_0^\infty x^{2n-1} e^{-x^2}\, dx\ ,$$

where n = 1, 2, 3, ... What should you do if you do not see a parameter that you can use for differentiation?

Solution

$$\int_0^\infty x e^{-ax^2}\, dx = \frac{1}{-2a}\int_0^\infty -2ax e^{-ax^2}\, dx = \frac{1}{-2a} e^{-ax^2}\Big|_0^\infty = \frac{1}{2a}$$

$$\int_0^\infty x^3 e^{-ax^2}\, dx = \left[-\frac{d}{da}\right]\int_0^\infty x e^{-ax^2}\, dx = \left[-\frac{d}{da}\right]\frac{1}{2a} = \frac{1}{2a^2}$$

$$\int_0^\infty x^5 e^{-ax^2}\, dx = \left[-\frac{d}{da}\right]\int_0^\infty x^3 e^{-ax^2}\, dx = \left[-\frac{d}{da}\right]\frac{1}{2a^2} = \frac{2}{2a^3}$$

$$\int_0^\infty x^7 e^{-ax^2}\, dx = \left[-\frac{d}{da}\right]\int_0^\infty x^5 e^{-ax^2}\, dx = \left[-\frac{d}{da}\right]\frac{2}{2a^3} = \frac{3\cdot 2}{2a^4}$$

$$\int_0^\infty x^9 e^{-ax^2}\, dx = \left[-\frac{d}{da}\right]\int_0^\infty x^7 e^{-ax^2}\, dx = \left[-\frac{d}{da}\right]\frac{3\cdot 2}{2a^3} = \frac{4\cdot 3\cdot 2}{2a^5}$$

Then we psyche out the pattern.

$$\int_0^\infty x^{2n-1}e^{-x^2}\,dx \underset{n=1}{\rightarrow} \int_0^\infty xe^{-ax^2}\,dx = \frac{1}{2a}$$

$$\int_0^\infty x^{2n-1}e^{-x^2}\,dx \underset{n=2}{\rightarrow} \int_0^\infty x^3e^{-ax^2}\,dx = \frac{1}{2a^2}$$

$$\int_0^\infty x^{2n-1}e^{-x^2}\,dx \underset{n=3}{\rightarrow} \int_0^\infty x^5e^{-ax^2}\,dx = \frac{2}{2a^3}$$

$$\int_0^\infty x^{2n-1}e^{-x^2}\,dx \underset{n=4}{\rightarrow} \int_0^\infty x^7e^{-ax^2}\,dx = \frac{3\cdot 2}{2a^4}$$

$$\int_0^\infty x^{2n-1}e^{-x^2}\,dx \underset{n=5}{\rightarrow} \int_0^\infty x^9e^{-ax^2}\,dx = \frac{4\cdot 3\cdot 2}{2a^5}$$

$$\int_0^\infty x^{2n-1}e^{-ax^2}\,dx = \frac{(n-1)!}{2a^n}$$

Now set $a = 1$.

$$\boxed{\int_0^\infty x^{2n-1}e^{-x^2}\,dx = \frac{(n-1)!}{2}}$$

SHORT-CUT NOTATION METHOD.

$$\int_0^\infty x^{2n-1}e^{-ax^2}\,dx = \left[-\frac{d}{da}\right]^{n-1}\int_0^\infty xe^{-ax^2}\,dx = \left[-\frac{d}{da}\right]^{n-1}\frac{1}{2a}$$

$$\int_0^\infty x^{2n-1}e^{-ax^2}\,dx = \frac{(n-1)!}{2a^n} \quad \Rightarrow \quad \boxed{\int_0^\infty x^{2n-1}e^{-x^2}\,dx = \frac{(n-1)!}{2}}$$

HW-B3. Integral with Quadratic and Linear Exponent. Use one of our tricks to evaluate the following integral where a > 0.

$$\int_{-\infty}^{\infty} xe^{-ax^2+bx}dx\ .$$

Solution

We have seen in class that

$$\int_{-\infty}^{\infty} e^{-ax^2+bx}\,dx = \sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}} \quad \text{for} \quad a > 0\ .$$

Then

$$\int_{-\infty}^{\infty} xe^{-ax^2+bx}dx = \frac{d}{db}\int_{-\infty}^{\infty} e^{-ax^2+bx}dx = \frac{d}{db}\left[\sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}}\right]$$

$$\int_{-\infty}^{\infty} xe^{-ax^2+bx}dx = \sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}}\,\frac{d}{db}\left(\frac{b^2}{4a}\right)$$

$$\boxed{\int_{-\infty}^{\infty} xe^{-ax^2+bx}dx = \sqrt{\frac{\pi}{a}}\,\frac{b}{2a}\,e^{\frac{b^2}{4a}}}$$