Physics I with Calculus, Prof. Ruiz (Doc), UNC-Asheville (1978-2021), doctorphys on YouTube
Chapter C. Projectiles and Circular Motion. Prerequisite: Calculus I. Corequisite: Calculus II.

C0. Tips on Studying Physics. Before starting this chapter, I would like to provide two important bits of advice given to me by one of my math teachers in high school. The teacher is Mr. Francis Cianfrani and he is pictured below looking on to one of his trigonometry students. I had him for math when I was a high school senior. The first semester included vector analysis.

Math Teacher Mr. Francis Cianfrani, Bishop Eustace Prep School, Pennsauken, NJ, USA, 1965

Here are the two excellent observations conveyed to me by Mr. Cianfrani.

1. Always Study Math with a Pencil in Hand. This advice is the best I can give you and it is especially true for this chapter and beyond. You have my notes and video lectures. For the best results, you should at some point write out all the steps in our course. This approach gives maximum benefit to you.

2. If You Understand a Given Level of Math, You Will Be Able to Understand the Next Level. This point involves prerequisites. For our course, you should have completed Calculus I and are taking Calculus II now. If that is the case, you should be fine handling the material in our course.

C1. Gravity Near Earth. Gravity near the Earth causes a mass to fall with constant acceleration if we neglect air resistance. We study gravity in detail later in our course. Here we are interested in the description of the motion without considering forces. In other words, we are continuing our coverage of kinematics. Remember, that kinematics deals with the description of motion and that dynamics introduces equations explaining the cause of the motion with forces.

The kinematic equations we have derived for constant acceleration in one dimension are listed in the table below.

$x = x_0 + v_0 t + \frac{1}{2}at^2$	no v
$v = v_0 + at$	no x
$x = x_0 + \frac{1}{2}(v_0 + v)t$	no a
$x = x_0 + \frac{v^2 - v_0^2}{2a}$	no t

Introducing $d = x - x_0$, these equations can also be written as shown in the next table.

$d = v_0 t + \frac{1}{2}at^2$	no v
$v = v_0 + at$	no d
$d = \frac{1}{2}(v_0 + v)t$	no a
$2ad = v^2 - v_0^2$	no t

I find the second table a little easier to memorize. You can often choose $x_0 = 0$ anyway.

The acceleration due to gravity near the surface of the Earth is

$$a = 9.8 \ \frac{\text{m}}{\text{s}^2},$$

in the downward direction. The convention is to call this acceleration g for gravity:

$$g = 9.8 \ \frac{\text{m}}{\text{s}^2}.$$

We make two observations at this point.

Observation 1. The acceleration g varies slightly depending on your location on the Earth. At the equator, due to the spinning Earth, the value is a little less compared to other locations. The *National Geodetic Survey (NGS)* of the *National Oceanic and Atmospheric Administration (NOAA)* uses field gravity data to interpolate in finding the gravity at your latitude, longitude, and elevation. Here is the link to their online surface gravity calculator. For Asheville, North Carolina, USA, where I live, the predictor calculator gives

$$g_{\text{Asheville}} = 9.79 \ \frac{\text{m}}{\text{s}^2} .$$

Physics texts usually go with two significant figures: $g = 9.8 \ \frac{\text{m}}{\text{s}^2}$.

Observation 2. All objects, regardless of weight, fall at the same acceleration. You can verify this fact by dropping two objects of different weights. If you use paper though, be sure to crumple it up. A nice demonstration is to first drop a ball and a flat sheet of paper, watching the ball reach the ground faster since air resistance slows the falling flat sheet. But then, crumpling up the paper shows them falling at the same rate. Aristotle (c. 350 BCE) got it wrong when he said heavier things fall faster and this error persisted for about 2000 years until Galileo (c. 1600) came along.

In science it is very important to verify results with real-world data. Let's do this by dropping a ball. I will drop a ball from a height $h = 77 \text{ inches} = 1.96 \text{ meters}$. I prefer, for this problem, to call the vertical variable z and pick z_0 to be zero at the top position of the ball about to be dropped. Also, I would like to measure positive distances downward from the drop point. The ball will accelerate at $g = 9.8 \ \frac{\text{m}}{\text{s}^2}$ as it falls the distance h . First write down the general formula

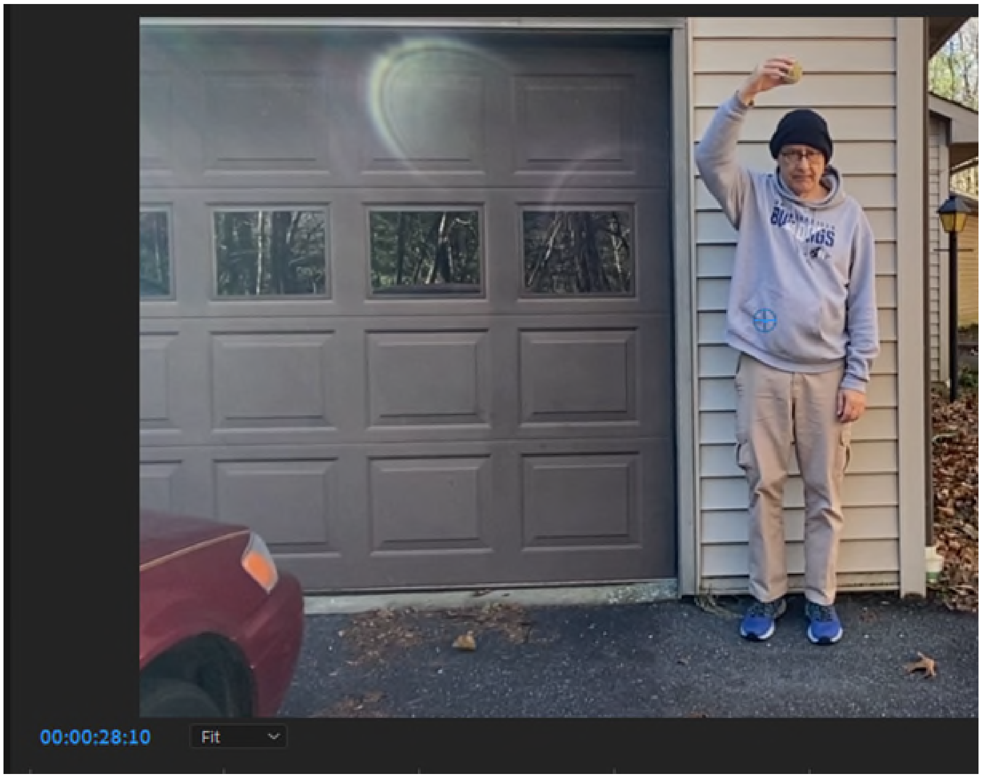
$$z = z_0 + v_0 t + \frac{1}{2} a t^2 .$$

Then set $z_0 = 0$. And since we are dropping the ball from rest, $v_0 = 0$. Finally, $a = +g > 0$ since down is positive and $z = h$ at the end of the fall. With these substitutions,

$$h = \frac{1}{2} g t^2 .$$

Alternatively, you could start with $d = v_0 t + \frac{1}{2} a t^2$ and simply set $v_0 = 0$ and $a = g$. Solving for the time of fall, we have $2h / g = t^2$ and

$$t = \sqrt{\frac{2h}{g}} == \sqrt{\frac{2 \cdot 1.96 \text{ m}}{9.8 \text{ m/s}^2}} = 0.63 \text{ s} = 0.6 \text{ s} .$$



Frame 10 past the 28-second mark on the video with the ball at h = 77 inches = 1.96 meters.

Frame 28 past the 28-second mark on the video with the ball hitting the ground.

The time of travel from the above stills obtained by video analysis is equal to 18 frames.

We need to convert frames to seconds using the frame rate of 30 frames per second.

$$18 \text{ frames} = 18 \text{ frames} \cdot \frac{1 \text{ second}}{30 \text{ frames}} = \frac{18}{30} \text{ s} = \frac{6}{10} \text{ s} = 0.6 \text{ s}$$

We are in agreement to one significant figure. It is best to go with one significant figure since there is a little uncertainty in the exact time of release and the exact time of reaching the ground.

Let's do a couple of kinematic problems with multiple parts to get practice using the formulas.

C2. Kinematics Problem: The Camaro. In our last chapter we discussed a "Camaro Stock Car" accelerating from rest to 300 km/h in 40 seconds. Here are three questions for this problem.

a) What is the average acceleration a in terms of g?
b) What is the distance traveled? You can take average acceleration as the constant a.
c) At what time is half the distance covered? Assume constant acceleration as above.

a. Average Acceleration. We know the initial velocity, the final velocity, and the time. So we pick the formula $v = v_0 + at$ and solve for the acceleration where $v_0 = 0$.

$$a = \frac{v - v_0}{t} = \frac{v}{t} = \frac{300 \text{ km/h}}{40 \text{ s}} = 7.5 \frac{\text{km}}{\text{h} \cdot \text{s}}$$

We continue in order to obtain the acceleration in terms of g.

$$a = 7.5 \frac{\text{km}}{\text{h} \cdot \text{s}} = 7.5 \frac{\text{km}}{\text{h} \cdot \text{s}} \cdot \frac{1000 \text{ m}}{1 \text{ km}} \cdot \frac{1 \text{ h}}{3600 \text{ s}} = 2.08 \frac{\text{m}}{\text{s}^2}$$

$$a = 2.08 \frac{\text{m}}{\text{s}^2} \cdot \frac{1 \text{ g}}{9.8 \text{ m/s}^2} = 0.21 \text{ g}$$

b. Distance Traveled. Since we know the initial and final velocities as well as the average acceleration, we use $2ad = v^2 - v_0^2$. I prefer to work in meters/second. Therefore, we first convert the speed:

$$v = 300 \frac{\text{km}}{\text{h}} = 300 \frac{\text{km}}{\text{h}} \cdot \frac{1000 \text{ m}}{1 \text{ km}} \cdot \frac{1 \text{ h}}{3600 \text{ s}} = 83.33 \frac{\text{m}}{\text{s}}.$$

I always keep extra significant figures and round off last. I consider the 300 km/h as 300. km/h, i.e., with three significant figures as our data from the last chapter was pretty accurate. The amount of significant figures should be given in homework problems. If I were making this a homework problem, I would state that the Camaro accelerated to a final speed of 300. m/s, including that decimal point after the 300. Refer to the table in the last chapter to get a feeling of the accuracy of the Camaro data. With $v = 83.33\text{ m/s}$, $v_0 = 0$, and $a = 2.08\ \dfrac{\text{m}}{\text{s}^2}$,

$$d = \frac{v^2 - v_0^2}{2a} = \frac{v^2}{2a} = \frac{(83.33\text{ m/s})^2}{2\cdot(2.08\text{ m/s}^2)} = 1669\text{ m} = 1670\text{ m} = 1.67\text{ km}.$$

c. Time to Cover Half the Distance. We can use

$$d = v_0 t + \frac{1}{2}at^2,$$

where $v_0 = 0$, $a = 2.08\dfrac{\text{m}}{\text{s}^2}$, and $d = \dfrac{1}{2}(1669\text{ m}) = 834.5\text{ m}$. Then, $d = v_0 t + \dfrac{1}{2}at^2$ becomes

$$834.5\text{ m} = \frac{1}{2}(2.08\text{ m/s}^2)t^2.$$

$$t^2 = \frac{2\cdot(834.5\text{ m})}{2.08\text{ m/s}^2}\text{ s}$$

$$t = \sqrt{\frac{1669\text{ s}^2}{2.08}}$$

$$t = 28.3\text{ s}$$

By the way, this time is $100\%\cdot\dfrac{28.3\text{ s}}{40\text{ s}} = 70\%$ of the journey.

Note that you only need 30% to cover the second half because you are going faster and faster.

C3. Kinematics Problem: The Window. This problem is inspired from one I encountered many years ago early in my teaching and I found it very tricky. Here, we are going to construct the problem so we will know the solution from the start. Then we will strip away everything but the bare minimum and propose the question. Constructing physics problems can be an excellent learning experience.

We start by considering a ball that is going to be fired upward from the ground at 40 m/s. We will take the acceleration due to gravity (g = 9.8 m/s^2) to be rounded off generously as g = 10 m/s^2 . Then, many of the numbers come out nicely and we will be less distracted by arithmetic.

We can quickly construct a table of speeds for every second from $v = v_0 + at$, where we choose upward as the positive direction. Then, $v(t) = v_0 + at = v_0 - gt = 40 - 10t$ and results are below.

t (s)	0	1	2	3	4
v (m/s)	40	30	20	10	0

Note that since we chose up as positive, the acceleration is negative, i.e., $a = -g < 0$. At t = 4 s, the ball is momentarily at rest at the top of its path. Then it falls down gaining the same speeds in reverse.

For the distances, we can adapt $x = x_0 + v_0 t + \frac{1}{2}at^2$ by picking y for the vertical coordinate and letting the ground be the zero reference. Then,

$$y = v_0 t + \frac{1}{2}at^2 = 40t + \frac{1}{2}(-g)t^2$$

$$y = 40t + \frac{1}{2}(-10)t^2$$

$$y(t) = 40t - 5t^2 .$$

The y-values for each second are easily calculated from this equation $y(t) = 40t - 5t^2$.

$$y(0) = 40 \cdot 0 - 5 \cdot 0^2 = 0$$

$$y(1) = 40 \cdot 1 - 5 \cdot 1^2 = 40 - 5 = 35$$

$$y(2) = 40 \cdot 2 - 5 \cdot 2^2 = 80 - 5 \cdot 4 = 80 - 20 = 60$$

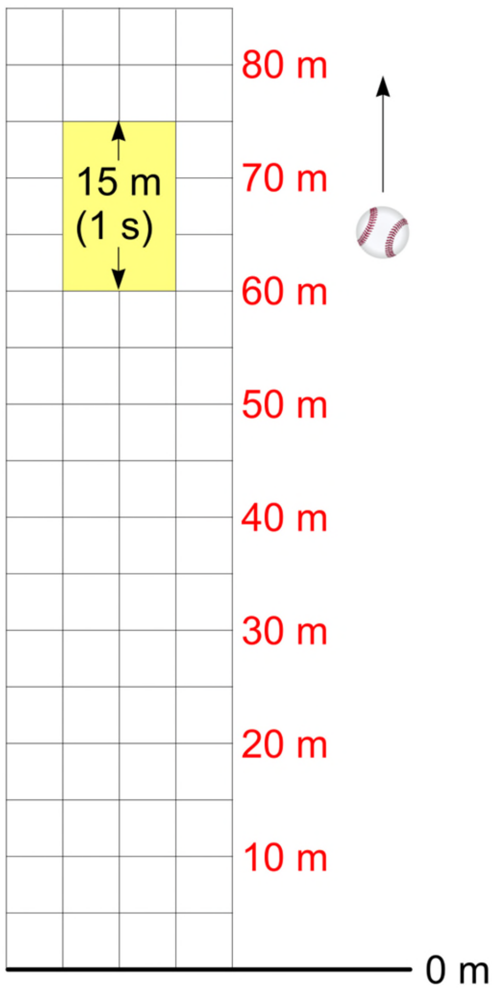
$$y(3) = 40 \cdot 3 - 5 \cdot 3^2 = 120 - 5 \cdot 9 = 120 - 45 = 75$$

$$y(4) = 40 \cdot 4 - 5 \cdot 4^2 = 160 - 5 \cdot 16 = 160 - 80 = 80$$

The table with the y-values is given below.

t (s)	0	1	2	3	4
v (m/s)	40	30	20	10	0
y (m)	0	35	60	75	80

For the next step in constructing our problem we draw a picture with a large window designated by the yellow rectangle. Between 2 s and 3 s the ball moves up from 60 m to 75 m. It takes one second for the ball to pass the window height on the way up. We will give this info in the problem and ask some questions.



Given: A ball is traveling upward having been shot up from the ground at 40 m/s. The ball is observed to take 1 s to pass a large 15-m window. **We DO NOT give the heights labeled red in the figure to the student.**

Questions. Use $g = 10 \frac{\text{m}}{\text{s}^2}$. Neglect air resistance.

(i) How far does the ball go above the window?
(ii) How far is the window bottom from the ground?

Solution. First, do not worry about answering the questions in order. What is easy to find first? The upward speed at the bottom of the window is easy to calculate using $d = v_0 t + \frac{1}{2}at^2$. Note that v_0 represents the velocity at the bottom of the window, which is the start for this segment of the problem. We know that $d = 15\text{ m}$, the time $t = 1\text{ s}$, and $a = -g = -10 \frac{\text{m}}{\text{s}^2}$. Therefore,

$$d = v_0 t + \frac{1}{2}at^2 \quad => \quad 15 = v_0 \cdot 1 - \frac{1}{2} \cdot 10 \cdot 1^2$$

$$15 = v_0 - 5$$

$$\boxed{v_0 = 20 \frac{\text{m}}{\text{s}}}$$

Refer to our previous table. It indicates $v(2\text{ s}) = 20 \frac{\text{m}}{\text{s}}$, although at this point, we (being the student) do not know about the 2 seconds. We just know that at the bottom of the window, the ball is traveling upward at $20 \frac{\text{m}}{\text{s}}$. We do not even know how far the window is above the ground.

We can now answer Question (i): How far does the ball go above the window? We use

$$2ad = v^2 - v_0^2,$$

where now d represents the distance from the bottom of the window to the maximum height of the ball. This new d is not the 15 m of before. We are starting a new problem, where d means something different. We also know that the ball at the top of the trajectory will have $v = 0$. The initial velocity for this problem is again though $v_0 = 20 \frac{\text{m}}{\text{s}}$. Plugging in the numbers,

$$2ad = v^2 - v_0^2$$

$$2(-10)\, d = 0^2 - 20^2$$

$$-20\, d = -400$$

$$d = \frac{400}{20}$$

$$\boxed{d = 20 \text{ m}}$$

The ball goes 20 m above the bottom of the window. So that means $20 \text{ m} - 15 \text{ m} = 5 \text{ m}$ above the top of the window. Consulting the table, you can see that the ball reaches up to 80 m, which is indeed 20 m from the bottom of the window. We have no idea at this point that the ball reaches 80 m above ground, but we do know the ball goes 5 m above the top of the window.

Distance Reached Above the Window: $\boxed{5 \text{ m}}$

Now consider Question (ii): How far is the bottom of the window from the ground? We can use

$$2ad = v^2 - v_0^2,$$

where $v_0 = 40 \frac{\text{m}}{\text{s}}$ (ground) and $v = 20 \frac{\text{m}}{\text{s}}$ at the bottom of the window. The distance d will be the distance from the ground to the bottom of the window. Substituting in the numbers,

$$2(-10)d = 20^2 - 40^2$$

$$-20d = 400 - 1600$$

$$-20d = -1200$$

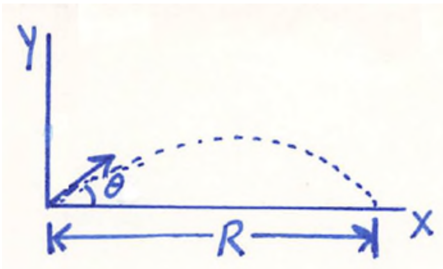
$$20d = 1200 \quad \Rightarrow \quad d = \frac{1200}{20} = \frac{120}{2}$$

$$\boxed{d = 60 \text{ m}}$$

Sure enough, from the table we see this is the correct answer. Since the ball goes up 20 m above the bottom of the window, we now know that the ball will reach a maximum height of 80 m.

C4. Projective Motion. Below is a diagram from my old notes dating back to 1978. We adapt $x = x_0 + v_0 t + \frac{1}{2}at^2$ for each dimension by writing

$$x = x_0 + v_{x0}t + \frac{1}{2}a_x t^2 \quad \text{and} \quad y = y_0 + v_{y0}t + \frac{1}{2}a_y t^2 .$$



For the x motion: $x_0 = 0$ $\quad v_{x0} = v_0 \cos\theta$ $\quad a_x = 0$
For the y motion: $y_0 = 0$ $\quad v_{y0} = v_0 \sin\theta$ $\quad a_y = -g$

We have chosen East as positive x and North as positive y. We also set the initial point as the origin $(x, y) = (0, 0)$. The initial velocity is v_0 and the angle θ is the initial angle. We neglect air resistance. There is no acceleration in the x-direction. The range is given by R.

Adapting the kinematic equations to our projectile leads to

$$\boxed{x = (v_0 \cos\theta)t} \quad \text{and} \quad \boxed{y = (v_0 \sin\theta)t - \frac{1}{2}gt^2}.$$

To arrive at the trajectory equation $y = y(x)$, use $x = (v_0 \cos\theta)t$ to solve for $t = \dfrac{x}{v_0 \cos\theta}$ and then substitute this t into the y equation.

$$y = (v_0 \sin\theta)t - \frac{1}{2}gt^2$$

$$y = (v_0 \sin\theta)\frac{x}{v_0 \cos\theta} - \frac{1}{2}g(\frac{x}{v_0 \cos\theta})^2$$

$$\boxed{y = (\tan\theta)x - \frac{gx^2}{2v_0^2 \cos^2\theta}}$$

We first make two algebraic observations that describe our projectile path in the above figure.

(i) The $-x^2$ means we have an upside-down parabola.
(ii) The x-term indicates that the parabola is shifted from the origin.

a) The Range. The range can be quickly found by setting y to zero. A student very early in my career pointed out to me that it is easier to work with y = y(t) for this part rather than y(x). Setting y to zero means the ball is on the ground. This condition is met at the beginning of the trajectory and at the end. Think of kicking an American football or a golf shot.

$$y = (v_0 \sin\theta)t - \frac{1}{2}gt^2 = 0$$

$$(v_0 \sin\theta)t - \frac{1}{2}gt^2 = 0$$

$$\left[(v_0 \sin\theta) - \frac{1}{2}gt\right]t = 0$$

By the way, do not divide both sides of the equation by t. If you do, you will lose one of the two solutions. The two solutions are

$$t = 0 \quad \text{and} \quad t = \frac{2v_0 \sin\theta}{g}.$$

These results are expected, the first solution being at the beginning of the trajectory and the second being at the end. For the range, we want the second solution, which we can call T, the total time for the trajectory. Plugging

$$t = T = \frac{2v_0 \sin\theta}{g} \text{ into } x = (v_0 \cos\theta)t \text{ gives the range:}$$

$$R = (v_0 \cos\theta)\frac{2v_0 \sin\theta}{g} = 2\cos\theta \sin\theta \frac{v_0^2}{g}.$$

Using the trig identity

$$2\cos\theta \sin\theta = \sin 2\theta \text{ , we arrive at}$$

$$\boxed{R = \frac{v_0^2}{g}\sin 2\theta}$$

The maximum range occurs when $\sin 2\theta$ maxes out at 1. The condition $\sin 2\theta = 1$ is met with

$$\theta = 45°.$$

b) The Maximum Height. The maximum height is also an interesting characteristic. The maximum height is reached at the halfway point, i.e., when

$$t_{1/2} = \frac{1}{2}T = \frac{1}{2}\frac{2v_0 \sin\theta}{g} = \frac{v_0 \sin\theta}{g} \; .$$

Substituting this time into the y(t) equation gives the maximum height

$$y_{\max} = (v_0 \sin\theta)t_{1/2} - \frac{1}{2}g(t_{1/2})^2$$

$$y_{\max} = (v_0 \sin\theta)\frac{v_0 \sin\theta}{g} - \frac{1}{2}g(\frac{v_0 \sin\theta}{g})^2$$

$$y_{\max} = \frac{(v_0 \sin\theta)^2}{g} - \frac{1}{2}\frac{(v_0 \sin\theta)^2}{g}$$

$$\boxed{y_{\max} = \frac{(v_0 \sin\theta)^2}{2g}} \quad \text{or} \quad \boxed{y_{\max} = \frac{v_0^2 \sin^2\theta}{2g}}$$

Let’s get some practice with max-min problems in calculus. You can also find this maximum y by taking the derivative $\frac{dy}{dt}$ and setting this derivative to zero. Remember, the derivative is the slope function. The slope is zero at the maximum point. Let’s do it.

$$y = (v_0 \sin\theta)t - \frac{1}{2}gt^2$$

$$\frac{dy}{dt} = (v_0 \sin\theta) - \frac{1}{2}g(2t) = 0$$

$$(v_0 \sin\theta) - gt = 0$$

$$v_0 \sin\theta = gt$$

$$t = \frac{v_0 \sin\theta}{g}$$

This value for t is the one we found before by taking one half of the total time of the trajectory.

Another cool idea from calculus is that if you take the second derivative, you can tell if the point of zero slope corresponds to a minimum or maximum. If the second derivative is positive you have a minimum as the slope near a minimum increases. Remember this rule: positive means it "holds water" and therefore the shape is a valley. A negative means you dump water and the extremum is a maximum. Let's take the first derivative again and then proceed to take the second derivative.

$$y = (v_0 \sin\theta)t - \frac{1}{2}gt^2$$

$$\frac{dy}{dt} = (v_0 \sin\theta) - \frac{1}{2}g(2t) = v_0 \sin\theta - gt$$

$$\frac{d^2y}{dt^2} = -g < 0$$

The slope is decreasing everywhere. Since $\frac{d^2y}{dt^2} < 0$, we do not "hold water" and the extremum is a maximum. Now it is time to do a trajectory problem with some numbers.

C5. A Baseball Problem. For a typical home run, $\theta = 30°$ and $v_0 = 100\ \frac{\text{mi}}{\text{h}}$.

Albert Pujols Home Run
Los Angeles Angels
September 8, 2014
Photo by Erik Drost
[Creative Commons License CC-BY-2.0]

For this problem, neglect air resistance and use the following values: $\theta = 30.0°$, $v_0 = 100.0\ \frac{\text{mi}}{\text{h}}$, $y_0 = 2.6\ \text{ft}$, and $g = 9.8\frac{\text{m}}{\text{s}^2} = 32\frac{\text{ft}}{\text{s}^2}$.

(a) What is the range of the ball in feet and in meters? Note that your answer will be much longer than real life since **air resistance shortens the range to a great degree for a fast-moving ball!**

(b) Will the ball clear an $18\ \text{ft}$ fence a distance $340\ \text{ft}$ away? If so, by how much in ft and m?

(c) How long does the ball remain in the air?

Solution. The relevant equation is listed below, where we include a $y_0 \neq 0$.

$$y = y_0 + (\tan\theta)x - \frac{gx^2}{2v_0^2\cos^2\theta}$$

We need to decide which units to work in. Since this problem is a baseball problem in the United States, we will start with the British system using feet.

$$v_0 = 100.0\ \frac{\text{mi}}{\text{h}} = 100.0\ \frac{\text{mi}}{\text{h}} \cdot \frac{5280\ \text{ft}}{1\ \text{mi}} \cdot \frac{1\ \text{h}}{3600\ \text{s}} = 146.67\ \frac{\text{ft}}{\text{s}}$$

Note that we always keep extra significant figures and round off last. Otherwise, during the several steps, we might "round off our answer away"!

A cool conversion to remember in order to quickly convert miles per hour to feet per second follows from considering a speed of $60\ \frac{\text{mi}}{\text{h}}$. Watch!

$$60.00\ \frac{\text{mi}}{\text{h}} = 60.00\ \frac{\text{mi}}{\text{h}} \cdot \frac{5280\ \text{ft}}{1\ \text{mi}} \cdot \frac{1\ \text{h}}{3600\ \text{s}} = 88.00\ \frac{\text{ft}}{\text{s}}$$

If you remember this trick, then you can do the following.

$$v_0 = 100.0\ \frac{\text{mi}}{\text{h}} = 100.0\ \frac{\text{mi}}{\text{h}} \cdot \frac{88.0\ \text{ft/s}}{60.0\ \text{mi/h}} = 146.67\ \frac{\text{ft}}{\text{s}}$$

So it can be helpful to remember

$$\boxed{60\ \frac{\text{mi}}{\text{h}} = 88\ \frac{\text{ft}}{\text{s}}}.$$

Summary of Input Parameters:

$\theta = 30.0°$, $v_0 = 146.67\ \frac{\text{ft}}{\text{s}}$, $y_0 = 2.6\ \text{ft}$, and $g = 32\frac{\text{ft}}{\text{s}^2}$.

Then the equation

$$y = y_0 + (\tan\theta)x - \frac{gx^2}{2(v_0\cos\theta)^2}$$

becomes

$$y = 2.6 + (\tan 30°)x - \frac{32x^2}{2\cdot(146.67\cos 30°)^2},$$

$$y(x) = 2.6 + 0.57735x - \frac{16x^2}{16{,}134} .$$

The range is found by setting $y = 0$. This time we are going to work with $y(x) = 0$ rather than $y(t) = 0$ as we did before.

$$0 = 2.6 + 0.57735x - \frac{16x^2}{16{,}134}$$

We have a quadratic equation $ax^2 + bx + c = 0$ where $a = -\frac{16}{16{,}134}$, $b = 0.57735$, and $c = 2.6$.

The solution to the quadratic equation is worked out below.

$$-\frac{16x^2}{16{,}134} + 0.57735x + 2.6 = 0$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-0.57735 \pm \sqrt{0.57735^2 - 4(-16/16{,}134)(2.6)}}{2(-16/16{,}134)}$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-0.57735 \pm \sqrt{0.333333 + 0.0103136}}{-0.0019834}$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-0.57735 \pm \sqrt{0.343647}}{-0.0019834}$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-0.57735 \pm 0.58621}{-0.0019834}$$

We want the positive solution since the range is positive. Therefore choose the case -0.58621.

$$R = \frac{-0.57735 - 0.58621}{-0.0019834} = \frac{0.57735 + 0.58621}{0.0019834} = \frac{1.1636}{0.0019834}$$

$$R = 586.7 = 590 \text{ ft}$$

$$\boxed{R = 590 \text{ ft}}$$

$$R = 586.7 \text{ ft} \cdot \frac{1 \text{ m}}{3.28 \text{ ft}} = 178.9 \text{ m}$$

$$\boxed{R = 180 \text{ m}}$$

Remember air resistance shortens the range to a great degree due to the fast ball speeds at the beginning of the path! The actual home run will have a range closer to about 400 ft or so.

(b) Will the ball clear a 18 ft (5.5 m) fence a distance 340 ft (104 m) away? If so, by how much?

$$y(x) = 2.6 + 0.57735x - \frac{16x^2}{16{,}134}$$

$$y(340) = 2.6 + 0.57735 \cdot 340 - \frac{16(340)^2}{16{,}134}$$

$$y(340) = 2.6 + 196.299 - 114.64 = 84.259\text{ ft} = 25.7\text{ m}$$

To 2 significant figures: $y(340) = 84\text{ ft} = 26\text{ m}$.

Distance above the fence: $84\text{ ft} - 18\text{ ft} = 66\text{ ft} = 20\text{ m}$.

(c) How long does the ball remain in the air? Use $x = (v_0 \cos\theta)t = 146.67\,\cos(30°)t$.

The range has been found to be $R = 586.7 = 590\text{ ft}$. When x corresponds to the range, the t variable beomes the total time of the trajectory.

Therefore the time of flight T is easily found.

$$t = \frac{x}{v_0 \cos\theta} = \frac{x}{146.67\cos 30°}$$

$$T = \frac{R}{146.67\cos 30°}$$

$$T = \frac{586.7}{146.67\cos 30°}$$

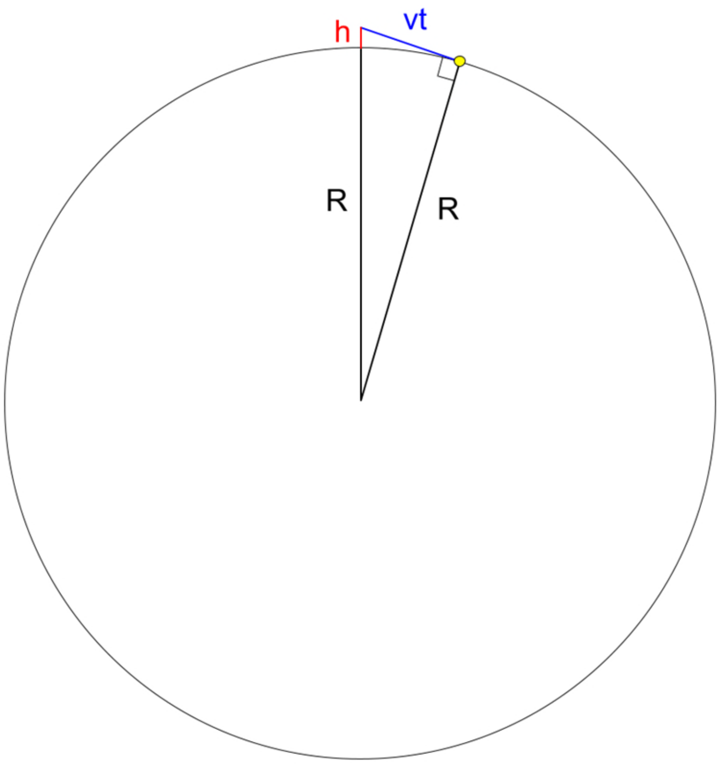
$$T = \frac{586.7}{127.0}$$

$$T = 4.62\text{ s}$$

$$\boxed{T = 5.0\text{ s}}$$

It is interesting that this time corresponds to an actual typical home-run time, which is about 4 to 6 seconds. The air resistance shortens the range, but also slows the ball down. These two effects sort of cancel and the time of travel is about the same for the no-air-resistance case and the real-life air-resistance case. When watching a ball game, the announcer will say something like the following, drawing out the repeated words "going" excitedly describing a home-run hit: "That ball is going, going, going. It is gone." And that takes about 5 seconds to say.

C6. Circular Motion. The yellow bead in the figure is traveling with constant speed v counter-clockwise along the circle. Although the speed is constant, the velocity is changing direction. Therefore, there is acceleration.



The yellow bead with the same direction would end up a height h above the circle at the top. But moving on the circle means it sort of "fell down" from that top position. **It falls towards the center.** Using the Pythagorean theorem,

$$(R+h)^2 = R^2 + (vt)^2 .$$

$$R^2 + 2Rh + h^2 = R^2 + (vt)^2$$

$$2Rh + h^2 = (vt)^2$$

Since $R >> h$, we have $2Rh >> h^2$ and can write $2Rh \approx (vt)^2$, which agrees more and more as the angle between the two radii decreases and approaches zero. Therefore, we obtain

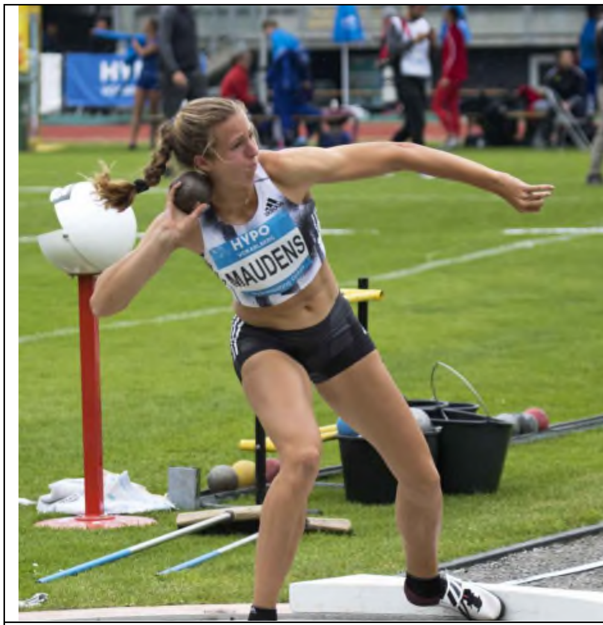
$$2Rh = (vt)^2 \quad \text{and} \quad 2Rh = v^2 t^2 .$$

Rearranging gives $h = \frac{1}{2}\frac{v^2}{R}t^2$. But this equation is of the form $h = \frac{1}{2}at^2$, where the acceleration is $a = \frac{v^2}{R}$. The **magnitude of the acceleration** for circular motion with constant speed v on a circle with radius R is then given by the following formula.

$$\boxed{a = \frac{v^2}{R}}$$

The direction of the acceleration vector is towards the center of the circle.

C7. Advanced Optional Bonus Section. The Best Angle When Launching Above Ground. We have seen that the best angle for the maximum range is $\theta = 45°$ when launching at the ground. When you consider hitting a baseball or hurling a shot put, the launch is above ground and other factors come into play such as the body's ability to achieve certain angles. So things are not so simple. The best shot putters in the world have a typical best angle around $\theta = 37°$. The most common angle for a home run in baseball is $\theta = 30°$. Our physics analysis below for launching above ground assumes that we can launch the projectile effectively at any angle, i.e., there are no other constraints such as the structure of the human body.

 	 "Chase Utley with the Philadelphia Phillies hitting a home run during a game against the Detroit Tigers on April 10, 2007." Wikimedia
Shot Put Photo Courtesy filip bossuyt, flickr Athlete: Hanne Maudens Event: Hypomeeting Götzis May 25, 2019 Creative Commons License	Baseball Photo Courtesy Googie Man, Wikipedia Athlete: Chase Utley Event: Phillies – Tigers Game, April 10, 2007 Creative Commons License

Lichtenberg and Wills were inspired by shot-put research to analyze from the point of view of physics the best angle when launching above the ground. See Hanne Maudens in the above photo about to make a shot put. The launch is above ground. Similarly for the baseball example in the right photo, Chase Utley hits the ball above the ground. Lichtenberg and Wills found that if you launch above the ground, the angle for optimum range is no longer 45° from the point of view of pure physics. Let's look at this publication. The reference is

D. B. Lichtenberg and J. G. Wills, "Maximizing the Range of the Shot Put," *American Journal of Physics* **46**, 546-549 (May 1978).

Lichtenberg and Wills first consider no air resistance. Then, for a shot put, they show that air resistance does not have a great effect since the ball does not travel at high speeds and does not travel extremely far. However, air resistance does have a large effect in other trajectories such as baseball. We will restrict ourselves to no air resistance and remember that for trajectories in general, air resistance matters.

There are three main parts for this analysis.

(1) Physics. Deriving the Range Equation $R = R(\theta)$ when the initial height is not zero.

(2) Calculus. Setting $\frac{dR(\theta)}{d\theta} = 0$, using rules for differentiation.

(3) Advanced Algebra. Solving the equation you find in Step (2). It will be very tricky!

This problem is an excellent one for gaining deep knowledge of physics, calculus, and solving a scary-looking algebra problem. It is well worth the time spent.

(1) Physics. We start with our basic equations

$$x = (v_0 \cos\theta)t \quad \text{and} \quad y = h + (v_0 \sin\theta)t - \frac{1}{2}gt^2,$$

where we have set $y_0 = h$, the height of the launch. Lichtenberg and Wills like to drop the "0" subscript on the velocity since they will only be discussing one velocity, the initial one.

$$x = (v\cos\theta)t \qquad y = h + (v\sin\theta)t - \frac{1}{2}gt^2$$

Note that long ago in this chapter we dropped the "0" subscript on the angle theta. Technically we should have θ_0, but since we are dealing with one angle here, there is no ambiguity. If you were to follow the angle during the trajectory, then we would need the subscript.

Let $t = T$ be the time of the flight, the time from the launch to the time of impact on the ground. Therefore, the object reaches the ground when $t = T$. At this time, $y(T) = 0$. The range $x(T) = R$ is the horizontal distance of the flight. We can write our equations with these parameters.

$$R = (v\cos\theta)T$$

$$0 = h + (v\sin\theta)T - \frac{1}{2}gT^2$$

In the publication, they then substitute $T = \frac{R}{v\cos\theta}$ from the first equation into the second equation.

$$0 = h + (v\sin\theta)\frac{R}{v\cos\theta} - \frac{1}{2}g(\frac{R}{v\cos\theta})^2$$

At this point, let me remind you that you are following a calculation in a published paper written for professional physicists! And you have just started studying physics.

$$0 = h + R\tan\theta - \frac{gR^2}{2v^2\cos^2\theta}$$

Now we have a surprise. Things are not so simple as they were before with $h = 0$. We are now faced with some unexpected algebra, a quadratic equation! The authors give the answer without showing the steps. It is typical in publications that all of the algebraic steps for the paper are not shown. It is expected that the reader fill in the steps with paper and pencil. So their next equation listed is

$$R = v^2\cos\theta\left[\sin\theta + (\sin^2\theta + \frac{2gh}{v^2})^{1/2}\right] / g .$$

Do you want to give it a shot – filling in the algebra? If so, pause to do the calculation. Then you can check your work with my algebra steps. I prefer to write the tangent out in terms of sine and cosine.

$$0 = h + R\tan\theta - \frac{gR^2}{2v^2\cos^2\theta}$$

$$0 = h + R\frac{\sin\theta}{\cos\theta} - \frac{gR^2}{2v^2\cos^2\theta}$$

Arrange things to get the equation in standard quadratic form: $ax^2 + bx + c = 0$.

$$-\frac{g}{2v^2\cos^2\theta}R^2 + \frac{\sin\theta}{\cos\theta}R + h = 0$$

Multiply by $-2v^2\cos^2\theta$ because I do not like denominators.

$$gR^2 - 2v^2\sin\theta\cos\theta R - 2v^2h\cos^2\theta = 0$$

We have a quadratic formula in R, where $a = g$, $b = -2v^2\sin\theta\cos\theta$, and $c = -2v^2h\cos^2\theta$.

The two solutions are

$$R = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2v^2\sin\theta\cos\theta \pm \sqrt{(-2v^2\sin\theta\cos\theta)^2 - 4g(-2v^2h\cos^2\theta)}}{2g}$$

$$R = \frac{2v^2 \sin\theta \cos\theta \pm \sqrt{(2v^2 \sin\theta \cos\theta)^2 + 4 \cdot 2gv^2 h \cos^2\theta}}{2g}$$

We next pull $(2v^2)^2$ out of the radical as $2v^2$. Note the appearance of $\frac{1}{v^2}$ in the second term under the square root sign in the equation below.

$$R = \frac{2v^2 \sin\theta \cos\theta \pm 2v^2 \sqrt{(\sin\theta \cos\theta)^2 + 2gh \cos^2\theta / v^2}}{2g}$$

$$R = \frac{v^2 \sin\theta \cos\theta \pm v^2 \sqrt{(\sin\theta \cos\theta)^2 + 2gh \cos^2\theta / v^2}}{g}$$

$$R = \frac{v^2 \sin\theta \cos\theta \pm v^2 \cos\theta \sqrt{\sin^2\theta + 2gh / v^2}}{g}$$

The angle θ ranges from 0° to 90°. To insure a positive R, we need the plus case.

$$R = \frac{v^2 \sin\theta \cos\theta + v^2 \cos\theta \sqrt{\sin^2\theta + 2gh / v^2}}{g}$$

$$R = \frac{v^2 \cos\theta \left[\sin\theta + \sqrt{\sin^2\theta + 2gh / v^2} \right]}{g}$$

We finally arrive at the equation in the publication. The algebra was an excellent workout for us.

$$\boxed{R = v^2 \cos\theta \left[\sin\theta + (\sin^2\theta + \frac{2gh}{v^2})^{1/2} \right] / g}$$

(2) Calculus. Next comes another excellent workout: a max-min problem in mathematics. We want to find the maximum range. So we set the derivative $\frac{dR}{d\theta} = 0$. Let $\alpha = \frac{2gh}{v^2}$ so we do not have to write down as much. Then $\frac{dR}{d\theta} = 0$ leads to

$$\frac{d}{d\theta}\{\frac{v^2}{g} \cos\theta \left[\sin\theta + (\sin^2\theta + \alpha)^{1/2} \right]\} = 0$$

We can divide out the $\frac{v^2}{g}$ factor. Then,

$$\frac{d}{d\theta}\{\cos\theta\left[\sin\theta + (\sin^2\theta + \alpha)^{1/2}\right]\} = 0 .$$

The calculus follows. The first step involves the product rule of differentiation. The general form for the product rule of differentiation is $\frac{d(fg)}{d\theta} = \frac{df}{d\theta}\cdot g + f\cdot\frac{dg}{d\theta}$.

$$\frac{d}{d\theta}\{\cos\theta\left[\sin\theta + (\sin^2\theta + \alpha)^{1/2}\right]\}$$

$$= \frac{d\cos\theta}{d\theta}\cdot\left[\sin\theta + (\sin^2\theta + \alpha)^{1/2}\right] + \cos\theta\frac{d}{d\theta}\left[\sin\theta + (\sin^2\theta + \alpha)^{1/2}\right] = 0$$

We continue using laws of differentiation.

$$-\sin\theta\cdot\left[\sin\theta + (\sin^2\theta + \alpha)^{1/2}\right] + \cos\theta\left[\frac{d\sin\theta}{d\theta} + \frac{d}{d\theta}(\sin^2\theta + \alpha)^{1/2}\right] = 0$$

$$-\sin\theta\left[\sin\theta + (\sin^2\theta + \alpha)^{1/2}\right] + \cos\theta\left[\cos\theta + \frac{1}{2}(\sin^2\theta + \alpha)^{1/2-1}\frac{d}{d\theta}(\sin^2\theta + \alpha)\right] = 0$$

$$-\sin\theta\left[\sin\theta + (\sin^2\theta + \alpha)^{1/2}\right] + \cos\theta\left[\cos\theta + \frac{1}{2}(\sin^2\theta + \alpha)^{-1/2}\cdot(2\sin\theta\frac{d\sin\theta}{d\theta} + \frac{d\alpha}{d\theta})\right] = 0$$

$$-\sin\theta\left[\sin\theta + \sqrt{\sin^2\theta + \alpha}\right] + \cos\theta\left[\cos\theta + \frac{1}{2}\frac{1}{\sqrt{\sin^2\theta + \alpha}}(2\sin\theta\cos\theta + 0)\right] = 0$$

$$-\sin\theta\left[\sin\theta + \sqrt{\sin^2\theta + \alpha}\right] + \cos\theta\left[\cos\theta + \frac{\sin\theta\cos\theta}{\sqrt{\sin^2\theta + \alpha}}\right] = 0$$

We have calculated $\frac{dR}{d\theta} = 0$ and found, writing the cosine term first, that

$$\boxed{\cos\theta\left[\cos\theta + \frac{\sin\theta\cos\theta}{\sqrt{\sin^2\theta + \alpha}}\right] - \sin\theta\left[\sin\theta + \sqrt{\sin^2\theta + \alpha}\right] = 0}.$$

Let’s take a break to stand back and look again at our plan.

(1) Physics. Deriving the Range Equation $R = R(\theta)$ when the initial height is not zero.

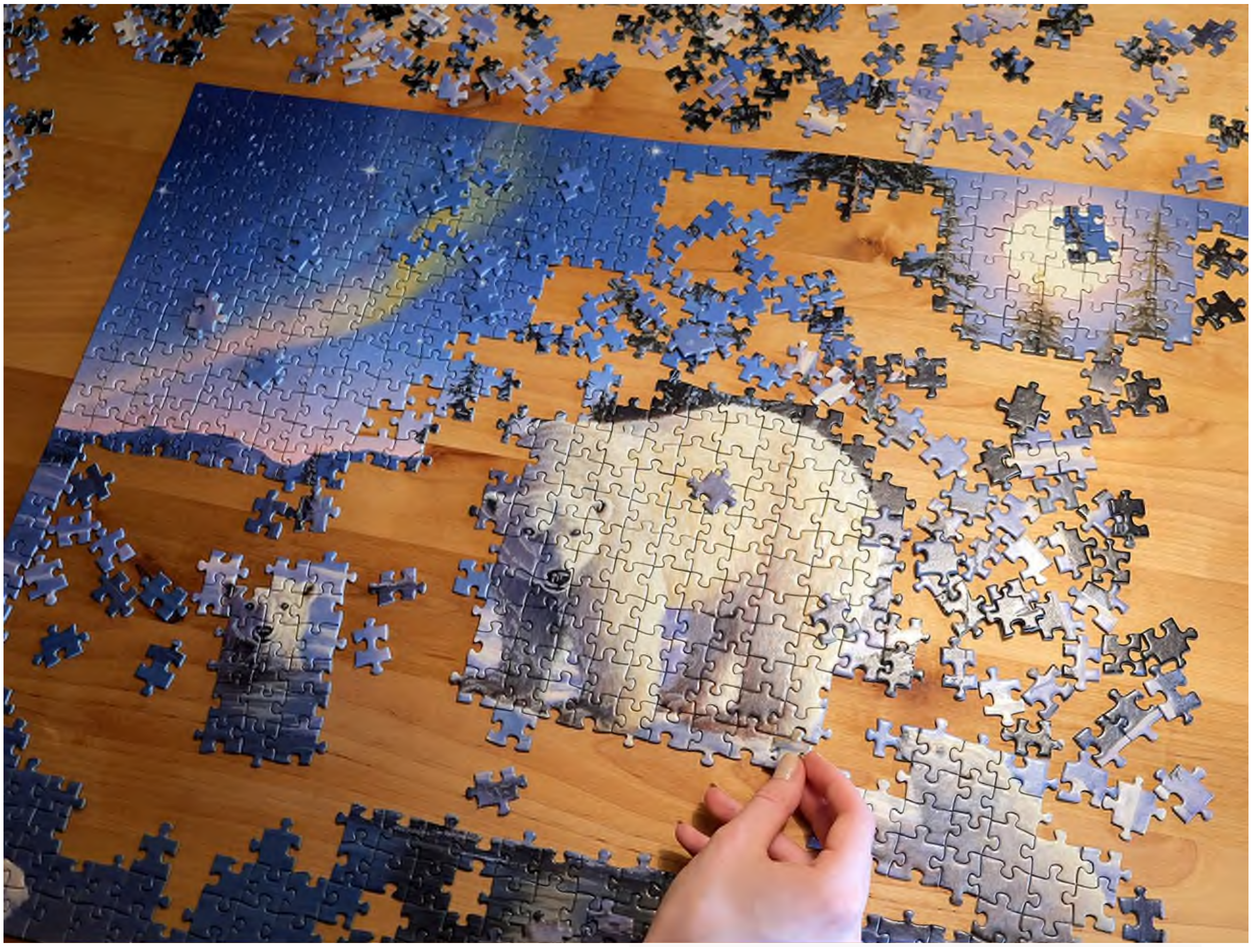
(2) Calculus. Setting $\dfrac{dR(\theta)}{d\theta} = 0$, using rules for differentiation.

(3) Advanced Algebra. Solving the equation you find in Step (2). It will be very tricky!

We have finished the first two steps, which involved much math. The third step will also conatin much math. We have to find the optimum angle and use this angle in the range equation to find the optimum range. So we can split step (3) into two parts.

(3a) Advanced Algebra. Solving the equation for the optimum angle θ.
(3b) Advanced Algebra. Using the optimum angle to find the optimum range R.

Physics often involves lots of math steps. Today, in practice, powerful tools like *Mathematica* can help with math. I tried some of these tools and found them helpful, but not completely useful in working through all the steps here. So, it is good to have mastery of algebra, trigonometry, and calculus. Physics or engineering is like a jigsaw puzzle with many pieces. If you find doing jigsaw puzzles fun, you may likewise find long and tedious math fun.



Jigsaw Puzzle Solving, kallerna, Wikimedia, Creative Commons Attribution-Share Alike 4.0

Summary. The range is

$$R = v^2 \cos\theta \left[\sin\theta + \sqrt{\sin^2\theta + \alpha} \right] / g \text{ , where } \alpha = \frac{2gh}{v^2} .$$

The condition for the maximum range is the angle you get when you solve the following equation.

$$\cos\theta \left[\cos\theta + \frac{\sin\theta\cos\theta}{\sqrt{\sin^2\theta + \alpha}} \right] - \sin\theta \left[\sin\theta + \sqrt{\sin^2\theta + \alpha} \right] = 0$$

You could include a subscript "m" for the angle to indicate that the angle is the one for the maximum range, but it is really not necessary since we are only considering one angle.

(3) Advanced Algebra. We need to solve the following equation

$$\cos\theta \left[\cos\theta + \frac{\sin\theta\cos\theta}{\sqrt{\sin^2\theta + \alpha}} \right] = \sin\theta \left[\sin\theta + \sqrt{\sin^2\theta + \alpha} \right],$$

$$\text{where } \alpha = \frac{2gh}{v^2} .$$

The above equation does not look friendly. How can you solve for the angle? A trick I learned in high school is that when you have a square root in there, you need to get the square root on one side of the equation and then square both sides. Now there a other approaches in solving this equation. I am going to take the straightforward way. It is the sure bet way to go, although, some of you may find shorter solutions. I will show you such a shorter solution later, one due to my friend Dr. Royce Zia.

To make the writing easier, define the following shortcut notation:

$$c = \cos\theta \text{ , } s = \sin\theta \text{ , and } A = \sqrt{\sin^2\theta + \alpha} = \sqrt{s^2 + \alpha} .$$

Then our equation becomes

$$c\left[c + \frac{sc}{A} \right] = s\left[s + A \right].$$

We need A on one side of the equation in preparation for squaring it.

$$c^2 + \frac{c^2 s}{A} = s^2 + sA$$

Multiply through by A.

$$Ac^2 + c^2 s = As^2 + sA^2$$

Use $A = \sqrt{\sin^2\theta + \alpha} = \sqrt{s^2 + \alpha}$ to replace A^2 with $A^2 = s^2 + \alpha$.

$$Ac^2 + c^2 s = As^2 + s(s^2 + \alpha)$$

Get all A terms on the left side.

$$A(c^2 - s^2) + c^2 s = s^3 + s\alpha$$

$$A(c^2 - s^2) = s^3 + s\alpha - c^2 s$$

We can go ahead and square now in order to avoid dividing both sides by $(c^2 - s^2)$.

$$A^2(c^2 - s^2)^2 = (s^3 + s\alpha - c^2 s)^2$$

Remember that $A = \sqrt{\sin^2\theta + \alpha} = \sqrt{s^2 + \alpha}$ and $A^2 = s^2 + \alpha$. Substituting $A^2 = s^2 + \alpha$,

$$(s^2 + \alpha)(c^2 - s^2)^2 = (s^3 + s\alpha - c^2 s)^2$$

Work out the square on the left side first.

$$(s^2 + \alpha)(c^4 - 2c^2 s^2 + s^4) = (s^3 + s\alpha - c^2 s)^2$$

Now work out the square on the right side (three squares and 3 cross terms with 2 in front).

$$(s^2 + \alpha)(c^4 - 2c^2 s^2 + s^4) = s^6 + s^2\alpha^2 + c^4 s^2 + 2s^4\alpha - 2s^4 c^2 - 2\alpha s^2 c^2$$

Work out the left side.

$$(s^2 c^4 - 2c^2 s^4 + s^6) + (\alpha c^4 - 2\alpha c^2 s^2 + \alpha s^4) = s^6 + s^2\alpha^2 + c^4 s^2 + 2s^4\alpha - 2s^4 c^2 - 2\alpha s^2 c^2$$

Write all the “s” factors before the “c” factors and α before “s” and “c”.

$$s^2 c^4 - 2s^4 c^2 + s^6 + \alpha c^4 - 2\alpha s^2 c^2 + \alpha s^4 = s^6 + \alpha^2 s^2 + s^2 c^4 + 2\alpha s^4 - 2s^4 c^2 - 2\alpha s^2 c^2$$

The s^6 cancels on each side. That’s nice!

$$s^2c^4 - 2s^4c^2 + \alpha c^4 - 2\alpha s^2c^2 + \alpha s^4 = \alpha^2 s^2 + s^2c^4 + 2\alpha s^4 - 2s^4c^2 - 2\alpha s^2c^2$$

The s^2c^4 cancels on each side.

$$-2s^4c^2 + \alpha c^4 - 2\alpha s^2c^2 + \alpha s^4 = \alpha^2 s^2 + 2\alpha s^4 - 2s^4c^2 - 2\alpha s^2c^2$$

The $-2s^4c^2$ cancels on each side.

$$\alpha c^4 - 2\alpha s^2c^2 + \alpha s^4 = \alpha^2 s^2 + 2\alpha s^4 - 2\alpha s^2c^2$$

The $-2\alpha s^2c^2$ cancels on each side.

$$\alpha c^4 + \alpha s^4 = \alpha^2 s^2 + 2\alpha s^4$$

Things are definitely looking better. We need to get rid of the cosine term to have only sine functions as we are trying to solve for s. So we use $c^2 = 1 - s^2$. Then, $c^4 = (1-s^2)^2 = 1 - 2s^2 + s^4$ and we arrive at

$$\alpha(1 - 2s^2 + s^4) + \alpha s^4 = \alpha^2 s^2 + 2\alpha s^4$$

$$\alpha - 2\alpha s^2 + \alpha s^4 + \alpha s^4 = \alpha^2 s^2 + 2\alpha s^4$$

$$\alpha - 2\alpha s^2 + 2\alpha s^4 = \alpha^2 s^2 + 2\alpha s^4$$

The $2\alpha s^4$ cancels on each side.

$$\alpha - 2\alpha s^2 = \alpha^2 s^2$$

Be careful about dividing by α since $\alpha = \dfrac{2gh}{v^2} = 0$ when $h = 0$.

So instead, we gather everything on one side,

$$\alpha^2 s^2 + 2\alpha s^2 - \alpha = 0,$$

and factor

$$\alpha(\alpha s^2 + 2s^2 - 1) = 0.$$

Solutions are either $\alpha = 0$ or $\alpha s^2 + 2s^2 - 1 = 0$. We want the latter general solution.

$$\alpha s^2 + 2s^2 - 1 = 0$$

Solve for s.

$$(\alpha+2)s^2-1=0$$

$$(\alpha+2)s^2=1$$

$$s^2=\frac{1}{\alpha+2}$$

$\sin^2\theta=\frac{1}{\alpha+2}$ or $\sin\theta=\sqrt{\frac{1}{\alpha+2}}$

$$\boxed{\sin^2\theta=\frac{1}{\frac{2gh}{v^2}+2}}$$

Now for a shorter way, one shown to me by my physicist friend Dr. Royce Zia.

Start again from $\cos\theta\left[\cos\theta+\frac{\sin\theta\cos\theta}{\sqrt{\sin^2\theta+\alpha}}\right]=\sin\theta\left[\sin\theta+\sqrt{\sin^2\theta+\alpha}\right]$.

$$c\left[c+\frac{sc}{A}\right]=s\left[s+A\right]$$

Now notice you can get $\left[s+A\right]$ on the left side.

$$c^2\left[1+\frac{s}{A}\right]=s\left[s+A\right] \quad \Rightarrow \quad \frac{c^2}{A}\left[A+s\right]=s\left[s+A\right] \quad \Rightarrow \quad \frac{c^2}{A}\left[s+A\right]=s\left[s+A\right]$$

$$\frac{c^2}{A}=s \quad \Rightarrow \quad c^2=sA \quad \Rightarrow \quad c^4=s^2A^2$$

Now use $A=\sqrt{\sin^2\theta+\alpha}=\sqrt{s^2+\alpha}$ to get $c^4=s^2(s^2+\alpha)=s^4+s^2\alpha$.

Use $c^2=1-s^2$ to arrive at $(1-s^2)^2=s^4+s^2\alpha$,

$$1-2s^2+s^4=s^4+s^2\alpha \quad \Rightarrow \quad 1-2s^2=s^2\alpha \quad \Rightarrow \quad 1=2s^2+s^2\alpha \quad \Rightarrow \quad 1=s^2(2+\alpha)$$

$$s^2=\frac{1}{\alpha+2} \quad \Rightarrow \quad \sin^2\theta=\frac{1}{\alpha+2} \quad \Rightarrow \quad \sin^2\theta=\frac{1}{\frac{2gh}{v^2}+2}$$

Note that unlike the case where we launch from the ground, the angle now depends on several parameters. With $h = 0$ we get $\sin^2\theta = \frac{1}{2}$ leading to $\theta = 45°$. In the general case, the above equation depends on g, h, and v. That means different results on different planets, where g is not Earth gravity. **Note that for large speeds we also get an angle of 45°! See below.**

$$\lim_{v\to\text{large}} \sin^2\theta = \lim_{v\to\text{large}} \frac{1}{\frac{2gh}{v^2}+2} = \frac{1}{0+2} = \frac{1}{2} \quad => \quad \theta = 45°$$

Let's use our $\sin^2\theta = \frac{1}{\alpha+2}$ result in for the range, which will now be maxed out.

$$R = \frac{v^2}{g}\cos\theta\left[\sin\theta + \sqrt{\sin^2\theta + \alpha}\right]$$

We need $\cos\theta$, $\sin\theta$, and $\sin^2\theta$. These terms are

$$\sin^2\theta = \frac{1}{\alpha+2}, \ \sin\theta = \frac{1}{\sqrt{\alpha+2}},$$

and

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{1}{\alpha+2}} = \sqrt{\frac{(\alpha+2)-1}{\alpha+2}} = \sqrt{\frac{\alpha+1}{\alpha+2}}.$$

With the above substitutions,

$$R = \frac{v^2}{g}\cos\theta\left[\sin\theta + \sqrt{\sin^2\theta + \alpha}\right] = \frac{v^2}{g}\sqrt{\frac{\alpha+1}{\alpha+2}}\left[\frac{1}{\sqrt{\alpha+2}} + \sqrt{\frac{1}{\alpha+2}+\alpha}\right]$$

$$R = \frac{v^2}{g}\sqrt{\frac{\alpha+1}{\alpha+2}}\left[\frac{1}{\sqrt{\alpha+2}} + \sqrt{\frac{1+\alpha(\alpha+2)}{\alpha+2}}\right]$$

$$R = \frac{v^2}{g}\sqrt{\frac{\alpha+1}{\alpha+2}}\left[\frac{1}{\sqrt{\alpha+2}} + \sqrt{\frac{1+\alpha^2+2\alpha}{\alpha+2}}\right]$$

$$R = \frac{v^2}{g}\sqrt{\frac{\alpha+1}{\alpha+2}}\left[\frac{1}{\sqrt{\alpha+2}} + \sqrt{\frac{(\alpha+1)^2}{\alpha+2}}\right]$$

$$R = \frac{v^2}{g}\sqrt{\frac{\alpha+1}{\alpha+2}}\left[\frac{1}{\sqrt{\alpha+2}} + \frac{\alpha+1}{\sqrt{\alpha+2}}\right]$$

$$R = \frac{v^2}{g}\sqrt{\frac{\alpha+1}{\alpha+2}}\left[\frac{\alpha+2}{\sqrt{\alpha+2}}\right]$$

$$R = \frac{v^2}{g}\sqrt{\frac{\alpha+1}{\alpha+2}}\left[\sqrt{\alpha+2}\right]$$

$$R = \frac{v^2}{g}\sqrt{\alpha+1}$$

What a nice simplification! How do we get back to the best angle? Remember our equations from above:

$\sin^2\theta = \frac{1}{\alpha+2}$, $\sin\theta = \frac{1}{\sqrt{\alpha+2}}$, and $\cos\theta = \sqrt{\frac{\alpha+1}{\alpha+2}}$,

along with our definition $\alpha = \frac{2gh}{v^2}$.

Using $\frac{v^2}{g} = \frac{2h}{\alpha}$,

$$R = \frac{v^2}{g}\sqrt{\alpha+1} = \frac{2h}{\alpha}\sqrt{\alpha+1}\ .$$

It is convenient to have α in terms of $\sin\theta$. Start with $\sin^2\theta = \frac{1}{\alpha+2}$ and solve for α.

$$\sin^2\theta = \frac{1}{\alpha+2}$$

$$\alpha+2 = \frac{1}{\sin^2\theta}$$

$$\alpha = \frac{1}{\sin^2\theta} - 2$$

$$\alpha = \frac{1-2\sin^2\theta}{\sin^2\theta}$$

Since we are shooting for $R = \frac{2h}{\alpha}\sqrt{\alpha+1}$, we next find

$$\alpha+1 = \frac{1-2\sin^2\theta}{\sin^2\theta}+1 = \frac{(1-2\sin^2\theta)+\sin^2\theta}{\sin^2\theta} = \frac{1-\sin^2\theta}{\sin^2\theta} = \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\tan^2\theta}$$

$$\alpha+1 = \frac{1}{\tan^2\theta} \qquad \alpha = \frac{1}{\tan^2\theta}-1$$

For $R = \frac{2h}{\alpha}\sqrt{\alpha+1}$, we can use $\sqrt{\alpha+1} = \frac{1}{\tan\theta}$ and $\alpha = \frac{1}{\tan^2\theta}-1 = \frac{1-\tan^2\theta}{\tan^2\theta}$.

We are now ready!

$$R = \frac{2h}{\alpha}\sqrt{\alpha+1} \qquad \frac{1}{\alpha} = \frac{\tan^2\theta}{1-\tan^2\theta} \qquad \sqrt{\alpha+1} = \frac{1}{\tan\theta}$$

Then,

$$R = \frac{2h}{\alpha}\sqrt{\alpha+1} = 2h\cdot\frac{1}{\alpha}\cdot\sqrt{\alpha+1} = 2h\cdot\left[\frac{\tan^2\theta}{1-\tan^2\theta}\right]\cdot\frac{1}{\tan\theta}$$

$$R = 2h\cdot\left[\frac{\tan\theta}{1-\tan^2\theta}\right]$$

$$R = h\cdot\left[\frac{2\tan\theta}{1-\tan^2\theta}\right]$$

$$R = h\cdot\tan(2\theta)$$

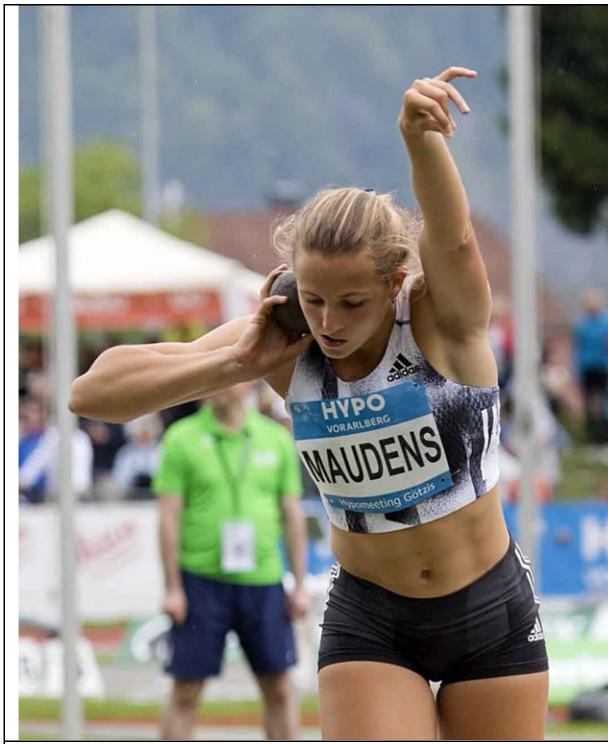
This result is surprisingly simple. The Lichtenberg and Wills paper does include the "m" subscript to reinforce the idea that we have the best angle, giving the maximum range.

$$\boxed{R_m = h\cdot\tan(2\theta_m)}$$

But since our context is clear, we just write

$$\boxed{R = h\cdot\tan(2\theta)}$$

This formula is super convenient because you can easily measure the launching height and range for a shot put. Let's input data for athlete Hanne Maudens. Since Maudens is world class, we can assume that she will be using the optimum angle, which may differ slightly from the pure physics analysis due to the structure of the body.



Shot Put Photo Courtesy filip bossuyt, flickr
Athlete: Hanne Maudens
Event: Hypomeeting Götzis May 25, 2019
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Athlete: Hanne Maudens
Born: March 12, 1997
Country: Belgium

Two Awards listed below.

Award: Bronze Medal
Category: Women's Heptathlon
Event: The 2016 World U20

Award: Gold Medal
Category: Women's Long Jump
Event: 2020 Belgian Indoor Athletics Championships

Shot Put Range: 13.95 m (45.77 ft)
Physical Height: 1.78 m (5' 10")

Since shot putters launch at a height greater than their standing height, we add 0.4 m = 40 cm (16 inches) to the physical height.

Then $h = 1.78 + 0.40 = 2.18 \text{ m}$ (7.2 ft) and $R = 13.95 \text{ m}$ (45.77 ft), giving

$$R = h \cdot \tan(2\theta) \text{ as}$$

$$13.95 = 2.18 \cdot \tan(2\theta) .$$

This result leads to

$$\tan(2\theta) = \frac{13.95}{2.18} = 6.399$$

$$2\theta = \tan^{-1}(6.399)$$

$$2\theta = 81.118°$$

$$\theta = 40.6°$$

As expected, the angle is less than 45°. We do not have the experimental measure of the angle Maudens used. But studies of many world class shot puts lead to an average near 37°. Researchers are still investigating why this angle is the best.

Here is how you can reflect on this answer and convince yourself that the angle needs to be less when you launch from a higher height. Suppose you were on a mountain so high that you could not hardly see the ground at the bottom of the valley? And you want to throw a ball as far as you can. What would your angle be? I think you would throw the ball straight out at $\theta = 0°$. So the higher you climb to launch the ball, the more you are going to decrease your angle down from 45°, until you get 0°. **What about if you were in a hole? Woud you use an angle greater than 45°?**

A panorama of Mount Kilimanjaro. Picture taken in Moshi, Tanzania
Courtesy Muhammad Mahdi Karim
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