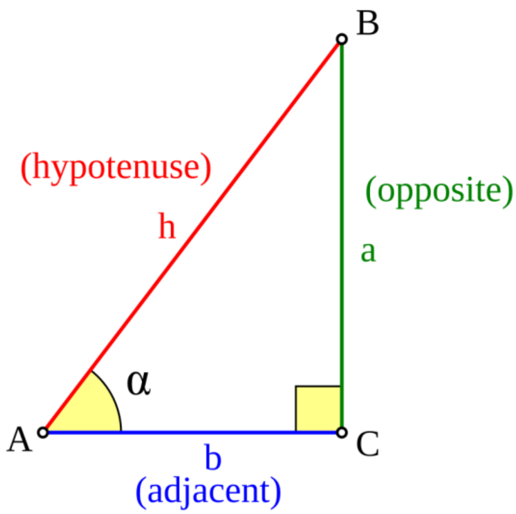
Basic Calculus. Prof. Ruiz (Doc), UNC-Asheville (1978-2021), DoctorPhys on YouTube.
Chapter B. Basic Trig. Prerequisites: High School Algebra and Trig.

In preparation of taking derivatives of trig functions, we provide here a quick review of trigonometry and associated functions. The definition of the sine of an angle can be given using a right triangle, i.e., a triangle where one of the angles is $90°$. See the figure below.



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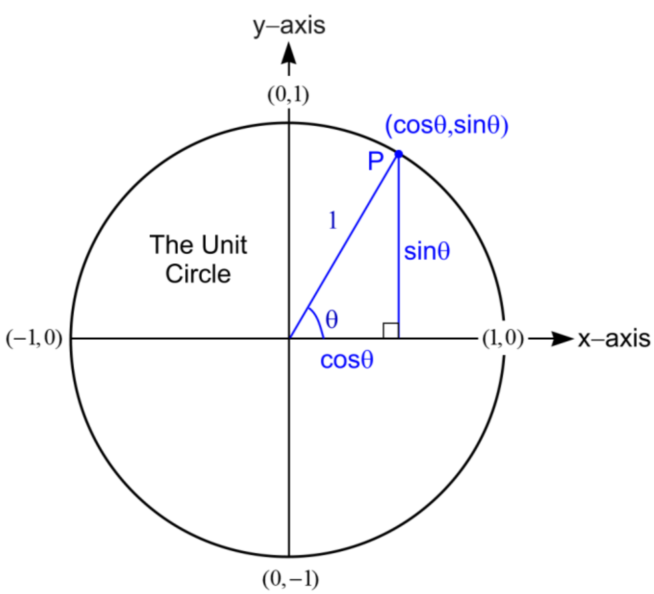
The sine of the angle α is equal to the ratio of the length of the opposite side divided by the length of the hypotenuse.

$$\sin\alpha = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{h}$$

The cosine of the angle α is equal to the ratio of the length of the adjacent side divided by the length of the hypotenuse.

$$\cos\alpha = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{h}$$



In the figure at the left we have the unit circle, i.e., a circle with radius 1. A right triangle is included where one vertex touches the center and another is on the circle. Using the same definitions above regarding adjacent and opposite sides, the (x, y) coordinates at point P are given by $(x, y) = (\cos\theta, \sin\theta)$. Note right off that the Pythagorean Theorem readily gives us an important identity relating $\cos\theta$ and $\sin\theta$.

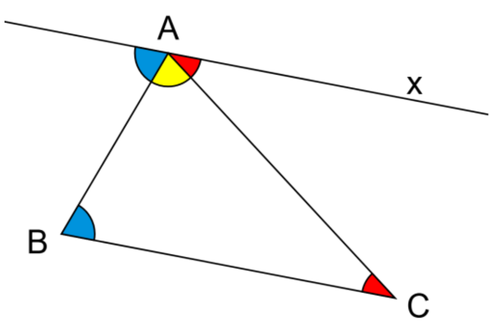
$$\cos^2\theta + \sin^2\theta = 1$$

The angle can be measured in degrees or radians. A right angle has a measure of $90°$, which is equivalent to $\frac{\pi}{2}$ radians. Think of a trip around the circle as sweeping out an angle of $360°$ or 2π radians. Note that the circumference of the unit circle is 2π since the circumference of a circle is $C = 2\pi r$, where the radius $r = 1$.

The definition of the tangent is $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$. There are six basic trig functions. These functions are defined in the table below.

cosine	sine	tangent	secant	cosecant	cotangent
$\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$	$\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$	$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$	$\sec\theta = \dfrac{1}{\cos\theta}$	$\csc\theta = \dfrac{1}{\sin\theta}$	$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

Many have trouble remember that the secant goes with the cosine since a natural naming convention would have been to assign csc with cos since they both begin with "c." One way to remember the correct one is that it is opposite to what one would expect. Or, use the mnemonic "Second Cosign" for having two people sign off on a contract. I actually had to cosign with my wife a couple of weeks ago (March 2025) during writing these notes for a FEMA application concerning private property Hurricane Helene (September 27, 2024) tree/debris removal.



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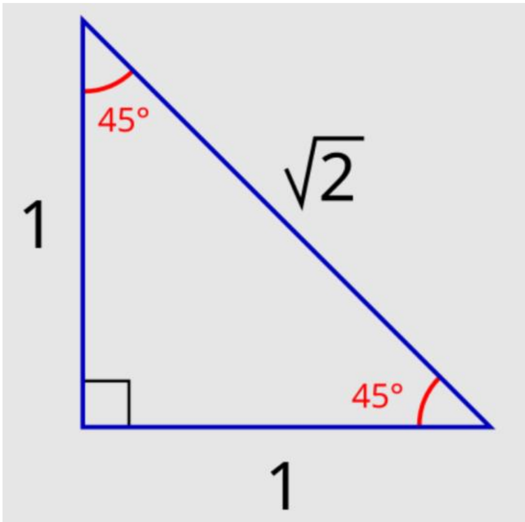
The figure at the left is a nice visual representation of the fact that the sum of the three angles in a triangle equals 180°. The straight line by definition is 180° and you can see that it can be divided into three angles equal to the three angles of the triangle. The blue pair and the red pair are called alternate interior angles. Let's determine some sines and cosines from special right triangles. The 45°-45°-90° triangle is perhaps the simplest right triangle. See the figure below. Note that the sum of the angles equals 180°.



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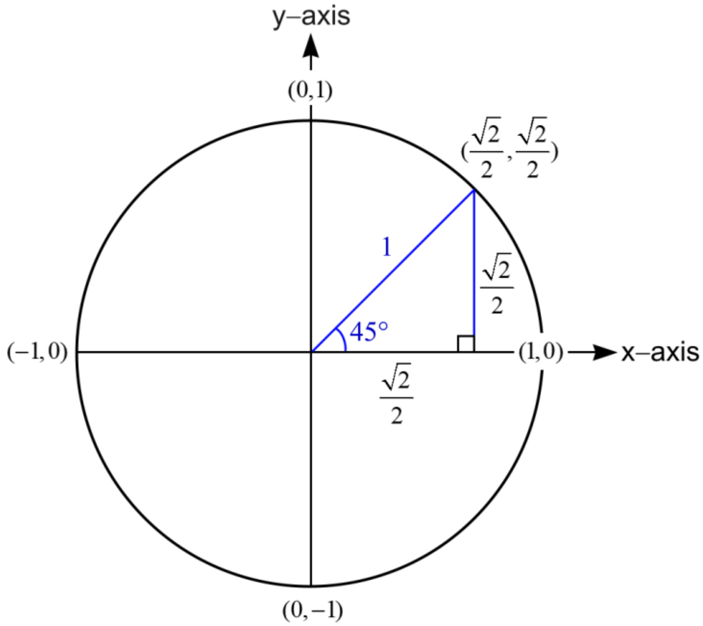
Choosing the horizontal and vertical sides to both be 1, the hypotenuse is $\sqrt{2}$ from the Pythagorean theorem. We can readily determine the cosine and sine for 45°.

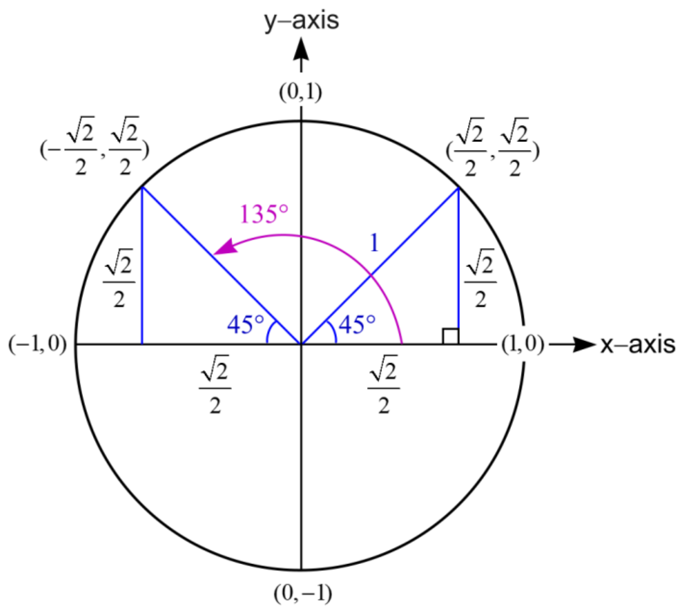
$$\cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \sin 45^\circ = \frac{\sqrt{2}}{2}$$

In terms of radians, $\cos\dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2}$ and $\sin\dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2}$.

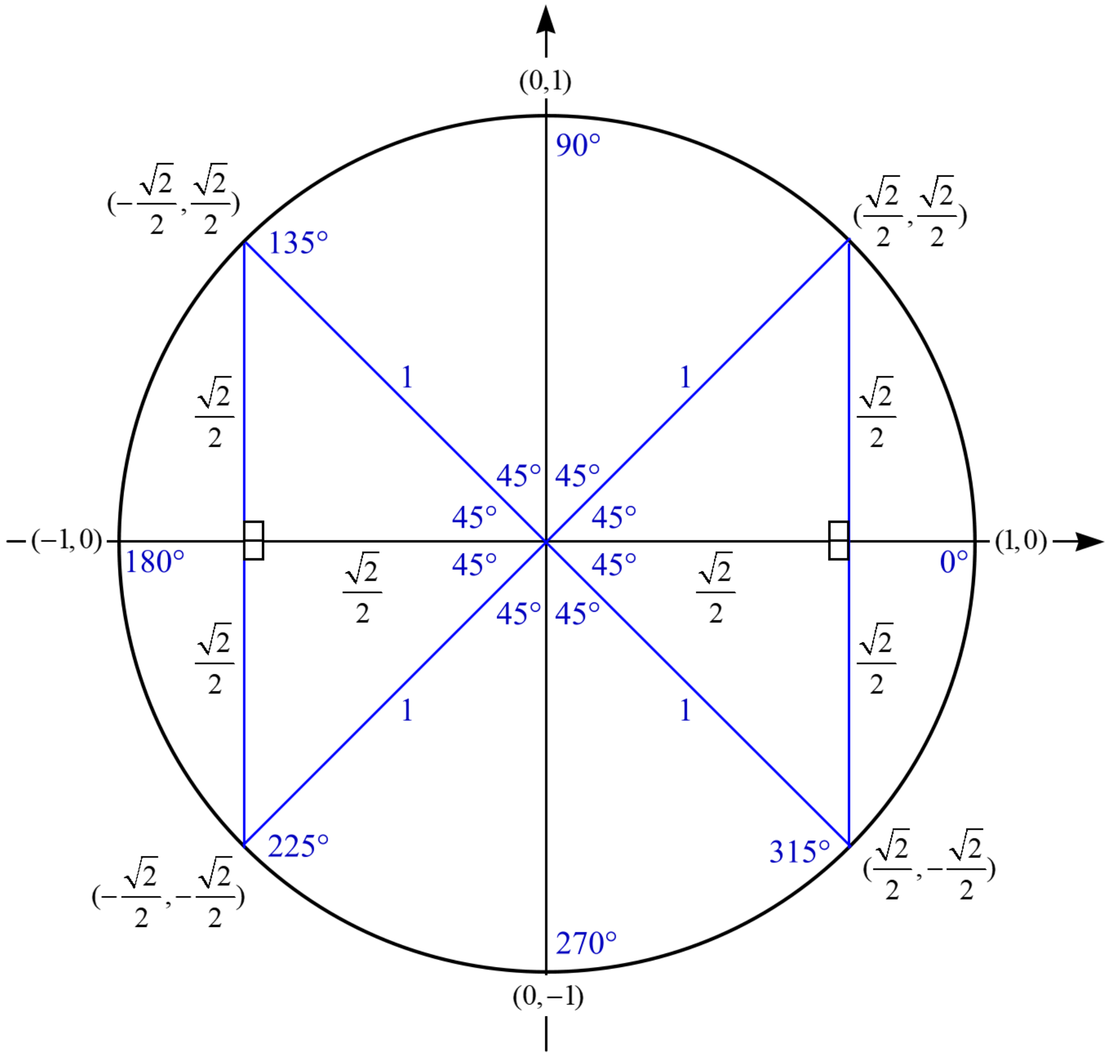
Now embed this right triangle inside the unit circle. The coordinates for the point on the circumference of the unit circle are $(\cos 45°, \sin 45°) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.



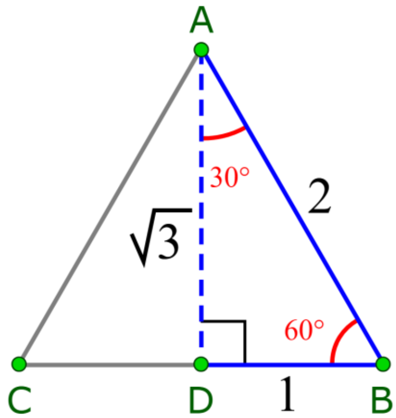
Note that we have the cosines and sines for an additional four points: $(\cos 0°, \sin 0°) = (1,0)$, $(\cos 90°, \sin 90°) = (0,1)$, $(\cos 180°, \sin 180°) = (-1,0)$, and $(\cos 270°, \sin 270°) = (0,-1)$. These values can simply be read off the figure. We can grab results for another angle by inserting another 45°-45°-90° triangle as shown below, namely, $(\cos 135°, \sin 135°) = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.



By cleverly inserting 45°-45°-90° triangles we an obtain a variety of related results. The cosine value is the left value in the ordered pair and the sine is the second value: $(x, y) = (\cos\theta, \sin\theta)$.



Next we consider the 30°-60°-90° triangle. We can determine the lengths by dividing an equilateral triangle into two 30°-60°-90° triangles. Label each side of the triangle with 2 units.



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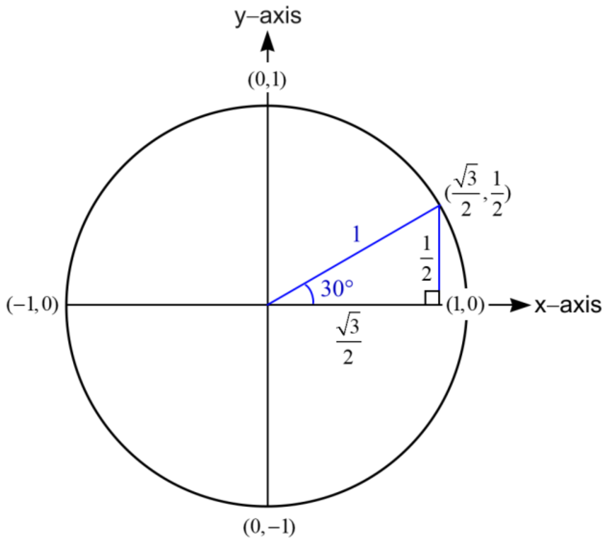
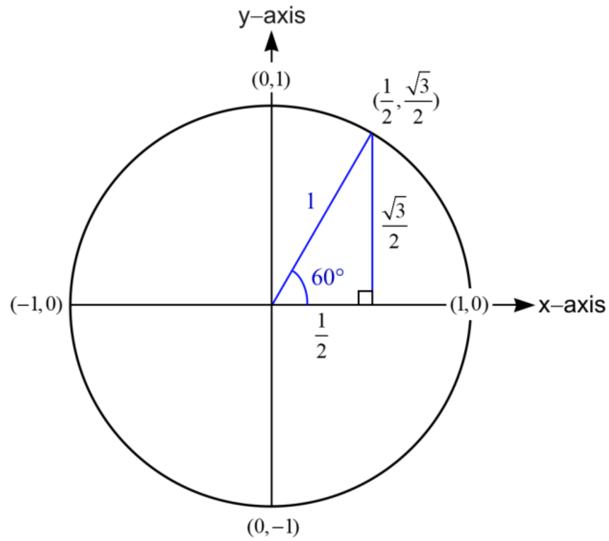
Then, DB is one-half a side. So, we take ½ of 2 and label it with a length of 1. We now apply Pythagorean's theorem to find AD.

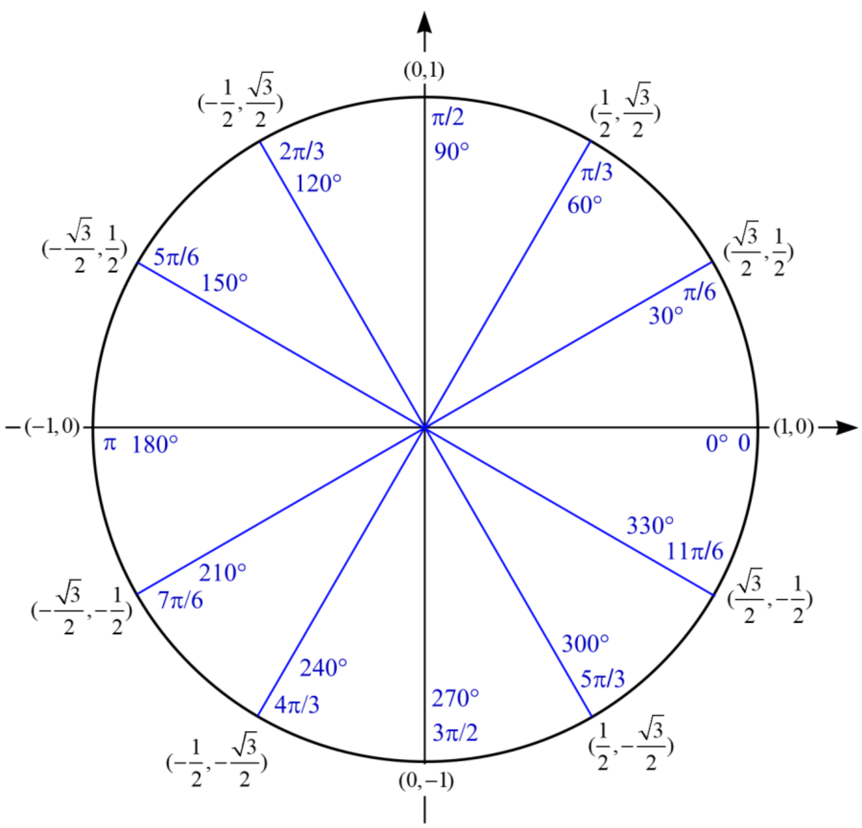
$$\overline{AD}^2 = \overline{AB}^2 - \overline{DB}^2 = 2^2 - 1^2 = 4 - 1 = 3$$

$$\overline{AD} = \sqrt{3}$$

We embed the 30°-60°-90° triangle in two basic ways below. We can now list some more cosine and sine values from the figure: $\cos 30° = \sin 60° = \dfrac{\sqrt{3}}{2}$ and $\sin 30° = \cos 60° = \dfrac{1}{2}$.







By symmetry, we can find numerous results, as shown in the figure at the left. Always remember that $(x, y) = (\cos\theta, \sin\theta)$.

We have also included the angles in radians along with degrees.

Study the figure to convince yourself of all of these possible applications of the 30°-60°-90° triangle.

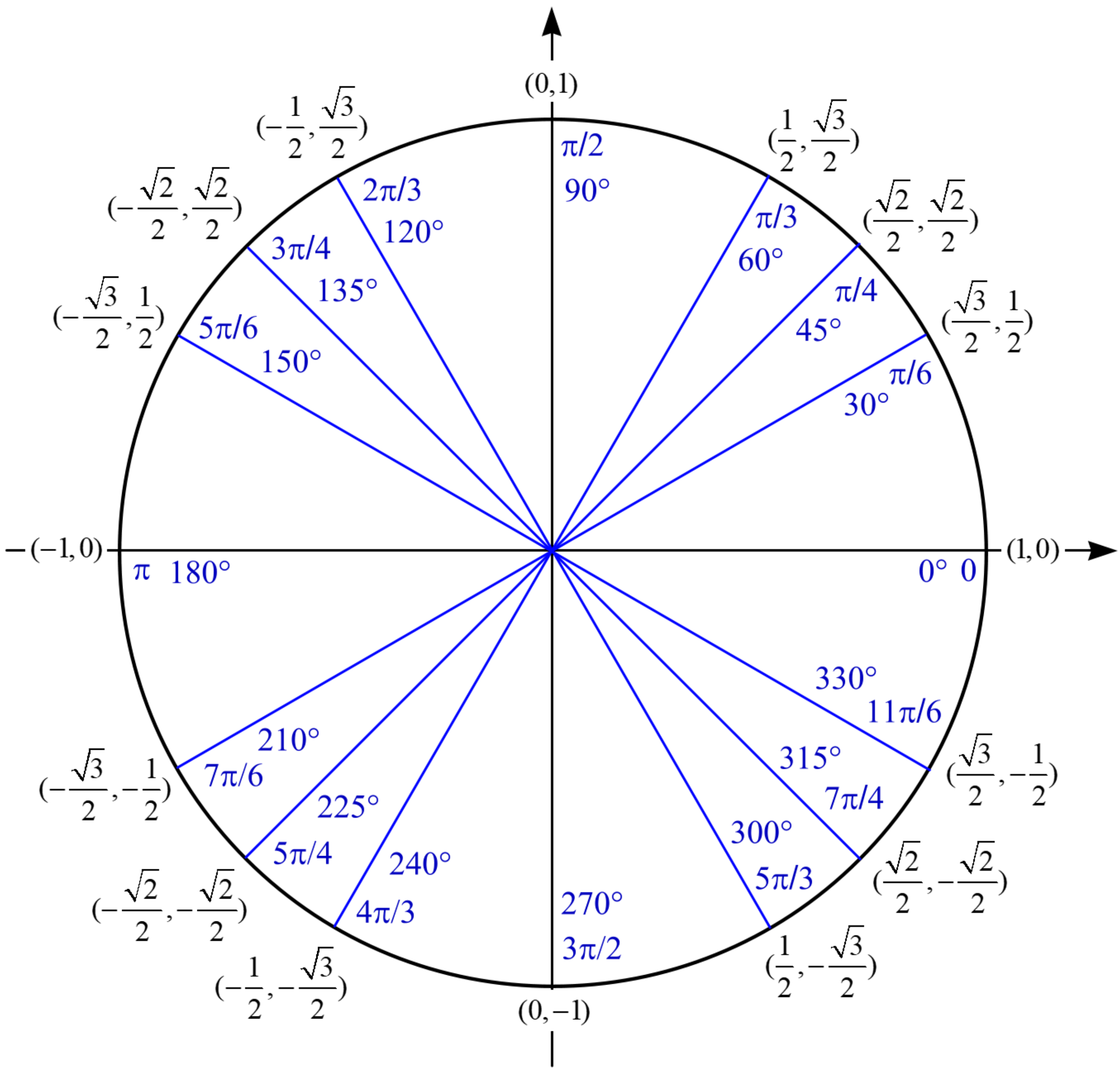
The tangents are readily found from

$$\tan\theta = \frac{\cos\theta}{\sin\theta} = \frac{y}{x}.$$

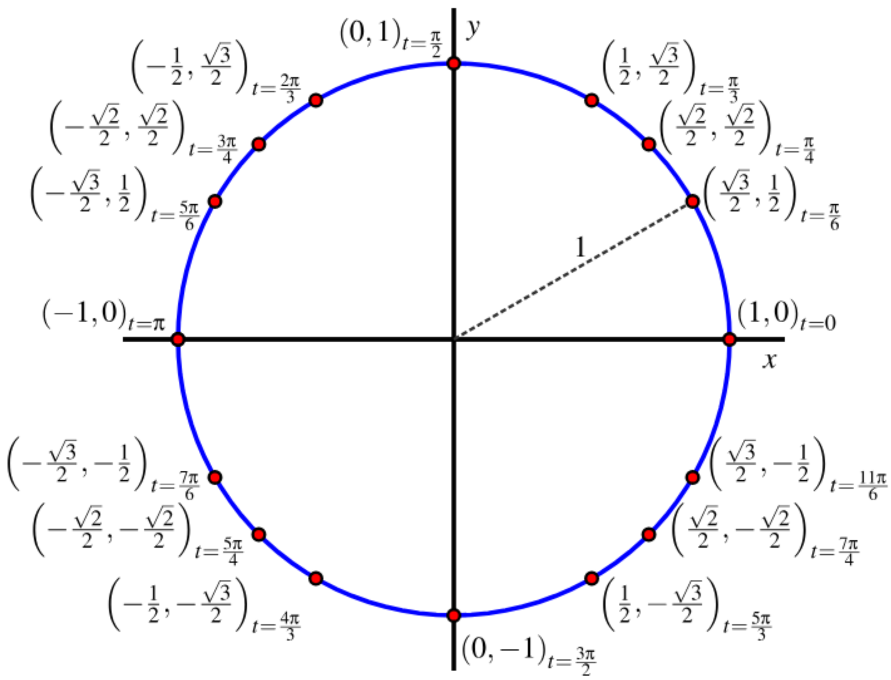
The figure below is the result of combined use of the 45°-45°-90° triangle and 30°-60°-90° triangle. The values for all six basic trig functions for many angles can be read off the figure, where $(\cos\theta, \sin\theta) = (x, y)$.

$$\cos\theta = x \qquad \sin\theta = y \qquad \tan\theta = \frac{y}{x}$$

$$\sec\theta = \frac{1}{x} \qquad \csc\theta = \frac{1}{y} \qquad \cot\theta = \frac{x}{y}$$



We would like to plot some the sine and cosine functions. We will focus on the sine function first.



Courtesy Matthew Boelkins
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The figure at the left is a reproduction of the figure on the previous page. The angle is represented by t given in radians.

For plotting, it is helpful to note that

$$\frac{1}{2} = 0.5,$$

$$\frac{\sqrt{2}}{2} = \frac{1.4142}{2} = 0.707$$

$$\frac{\sqrt{3}}{2} = \frac{1.732}{2} = 0.886.$$

The sine value for the red points in the above figure are plotted below.

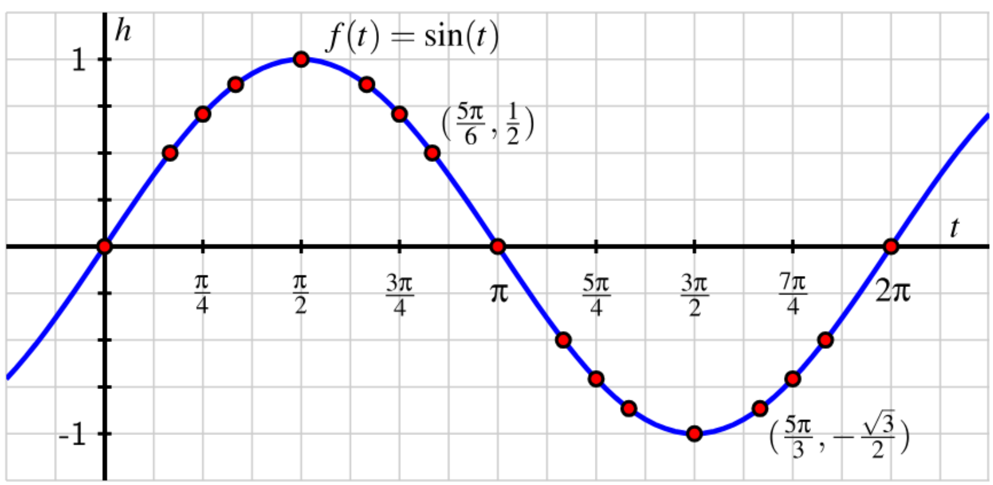
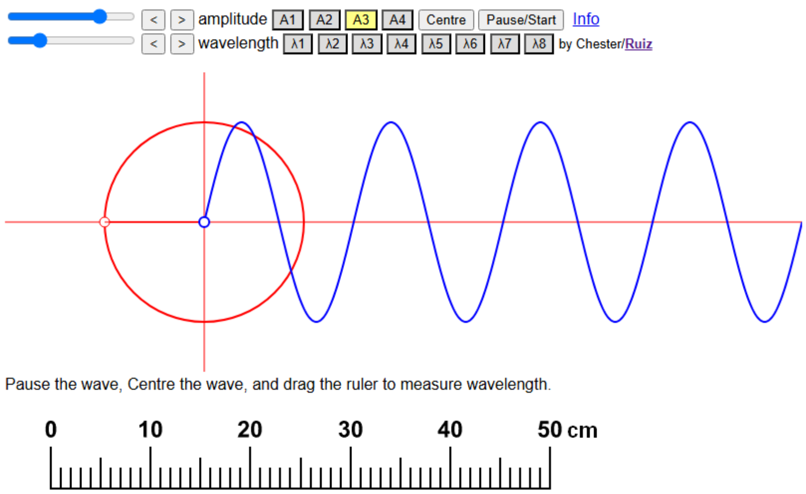


Figure Courtesy Matthew Boelkins, activecalculus.org, License: CC BY-SA 2.0.

The sine and cosine are intimately related to the circle, as we have seen. Below is a nice visualization of this relationship.



The bead in the left figure is located at the moment. You can take the radius of the circle to be unity. As this bead swings up and down, the various heights moment by moment give the sine: $y = \sin\theta$

For different radii R we can write

$$y = R\sin\theta .$$

In physics, we are fond of using A for amplitude: $y = A\sin\theta$.

Explore this application which I publish with a high school student. The link is https://mjtruiz.com/ped/circleSine/. At the time of the work my coauthor Ethan Chester was in high school. The publication reference is given below along with a link to the manuscript and the app.

Ethan Chester (Asheville High School Student) and Michael J. Ruiz, "HTML5 lab app relating circular motion to harmonic motion and the wave equation," *Physics Education* **55**, 013004 (January 2020). pdf and The App

θ (deg)	0	30	45	60	90	120	135	150	180	210	225	240	270	300
θ (rad)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$
$\sin t$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$
$\cos t$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$

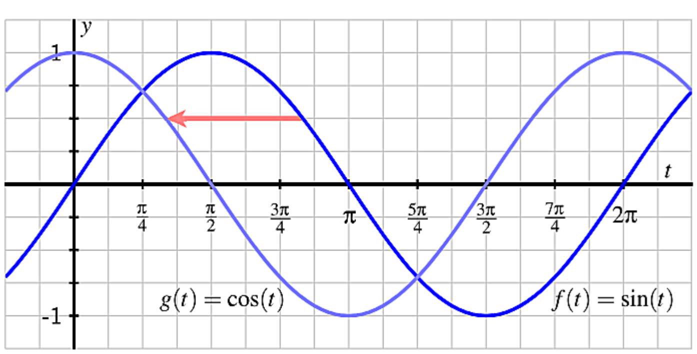
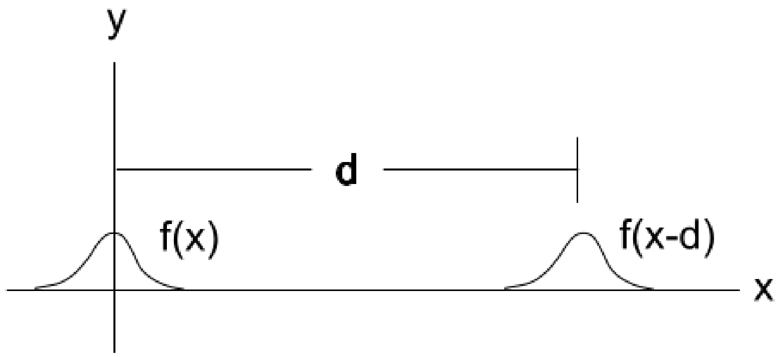


Figure Courtesy Matthew Boelkins, activecalculus.org, License: CC BY-SA 2.0. Change: Reversed direction of arrow.

If you shift the $\sin t$ values in blue in the table to the left by $\pi / 2$ you obtain $\cos t$. How do you shift a function to the left or right?

We demonstrate shifting a function by consider the function $f(x)$ which is a little mountain over the origin. Therefore, the function $f(x)$ has a peak at $x = 0$. Denote this by writing $f(0) = peak$. If we shift this function to the right by a distance d, then the new function $h(x)$ must be $h(x) = f(x-d)$. Here is how you can check this rule. Is the peak now at $x = d$? Does $h(d) = peak$? We work out the details below the figure.



$$h(x) \equiv f(x-d)$$

$$h(d) = f(d-d) = f(0) = peak$$

It checks out.

If we wanted to shift the peak to the left we would use $h(x) = f(x+d)$.

We summarize these two rules below.

shift $f(x)$ to the right by d, use $f(x-d)$,
shift $f(x)$ to the left by d, use $f(x+d)$..

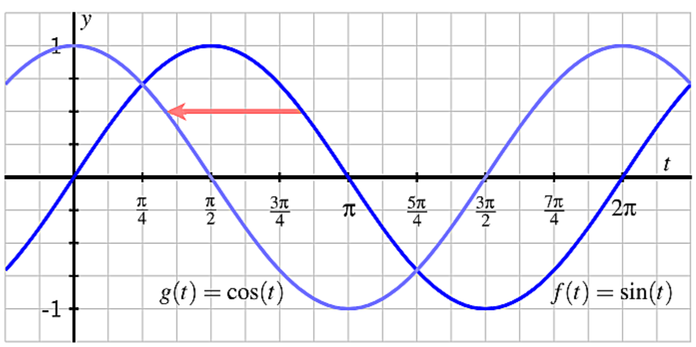


Figure Courtesy Matthew Boelkins, activecalculus.org, License: CC BY-SA 2.0. Change: Reversed direction of arrow.

In the figure, if you shift $f(t) = \sin t$ to the left by $\pi / 2$ to obtain $g(t) = \cos t$, we can write this relationship as

$$g(t) = \cos t = f(t + \frac{\pi}{2}) = \sin(t + \frac{\pi}{2}).$$

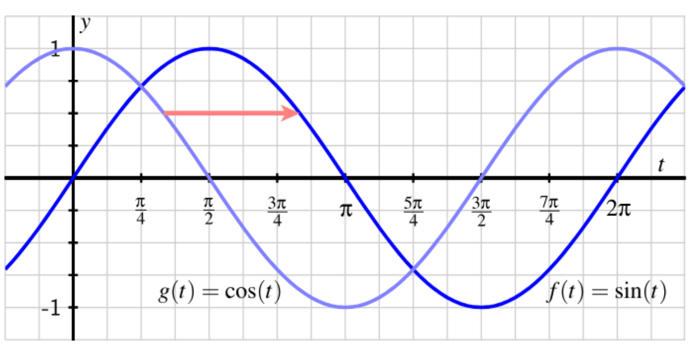


Figure Courtesy Matthew Boelkins, activecalculus.org, License: CC BY-SA 2.0

Now let's shift the cosine $g(t) = \cos t$ to the right by $\pi / 2$ to obtain $f(t) = \sin t$.

$$f(t) = \sin t = g(t - \frac{\pi}{2}) = \cos(t - \frac{\pi}{2})$$

Note that $\cos(-t) = \cos(t)$ as there is reflection symmetry about the vertical y-axis. We say that the cosine is an even function.

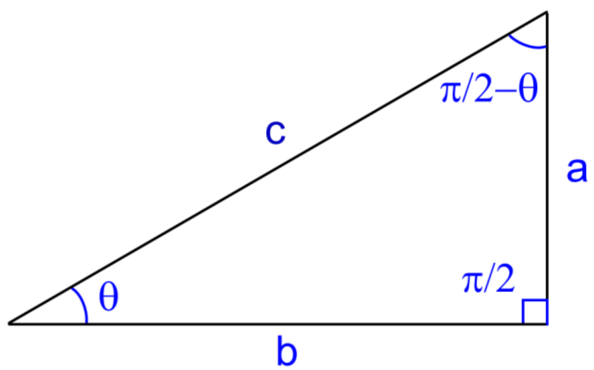
On the other hand, $\sin(-t) = -\sin(t)$, i.e., the sine function is an odd function. But since $\cos(-t) = \cos(t)$, we can write out equation $\sin t = \cos(t - \frac{\pi}{2})$ as

$$\sin t = \cos(\frac{\pi}{2} - t).$$

I will often use x as the input variable for a function.

$$\sin x = \cos(\frac{\pi}{2} - x)$$

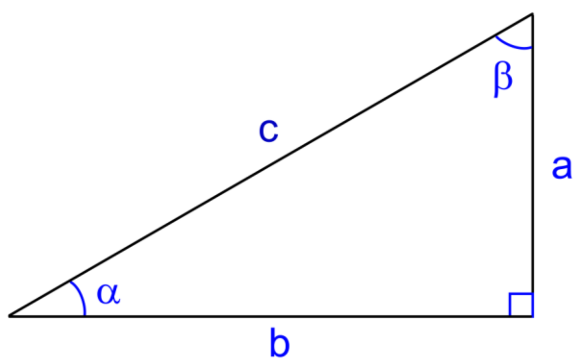
But we kind of already know this fact. See the triangle below, where I choose "b" for base and "a" for altitude.



The sum of the angles for the triangle must be π, i.e., $180°$. Since we have a right angle, i.e., one angle equals $\pi / 2$, the sum of the other two angles must be $\pi / 2$. From the definitions of the sine and cosine, we have

$$\sin\theta = \frac{b}{c} \quad \text{and} \quad \cos(\frac{\pi}{2} - \theta) = \frac{b}{c}.$$

Therefore, $\sin\theta = \cos(\frac{\pi}{2} - \theta)$.



Refer to the triangle at the left. I choose angle α to be opposite side a since each is the first letter in an alphabet, Greek and Latin respectively. Similarly, β and b are opposite each other as second letters in alphabets. Since,

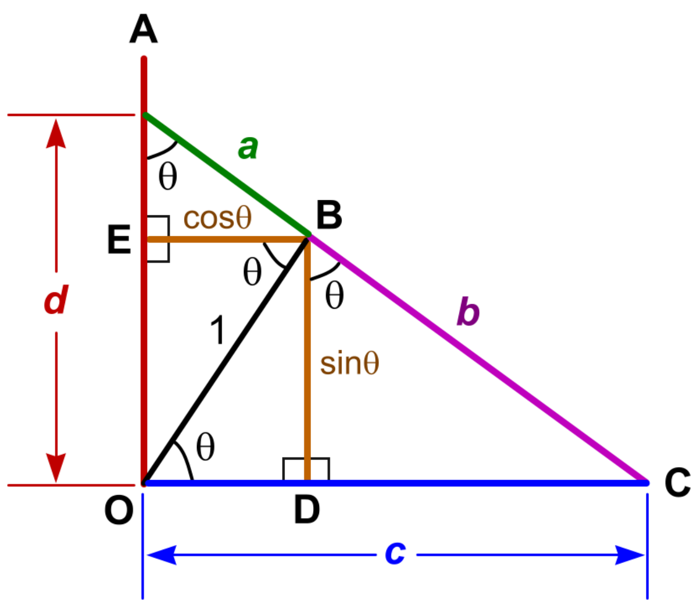
$$\alpha + \beta = \frac{\pi}{2}.$$

we say the angles are complementary. Another way to state $\sin\theta = \cos(\frac{\pi}{2} - \theta)$ is by

$$\sin\alpha = \cos\beta,$$

where α and b are complementary.

Finally, before leaving this chapter, I would like to give you a geometric visualization of all six trig functions. The secret is to use the following figure, which I adapted from an excellent figure for which I was inspired to make after seeing a similar figure at an MIT webpage.



In the figure at the left we show the standard $\cos\theta$ and $\sin\theta$ representations with the usual triangle ΔOBD. We then construct the similar triangle ΔABE to the north of ΔOBD and a similar triangle ΔBCD to the right of ΔOBD. If two triangles each contain a θ and a 90-degree angle, their third respective angles are equal and they are similar. Note also that ΔBCO is similar to ΔOBD. Here is how we indicate the similar triangles.

$$\Delta OBD \sim \Delta ABE \sim \Delta BCD \sim \Delta BCO$$

First focus on ΔOBD and ΔABE. The ratio of each hypotenuse to the side opposite θ must be equal.

$$\frac{1}{\sin\theta} = \frac{a}{\cos\theta} \quad => \quad a = \frac{\cos\theta}{\sin\theta} = \cot\theta \quad => \quad a = \cot\theta$$

Next consider ΔOBD and ΔBCD in order to find b. If we try our same strategy, we find

$$\frac{1}{\sin\theta} = \frac{b}{b\sin\theta} = \frac{1}{\sin\theta} \text{ and we discover nothing!}$$

But we can compare the hypotenuse to the adjacent side in each case.

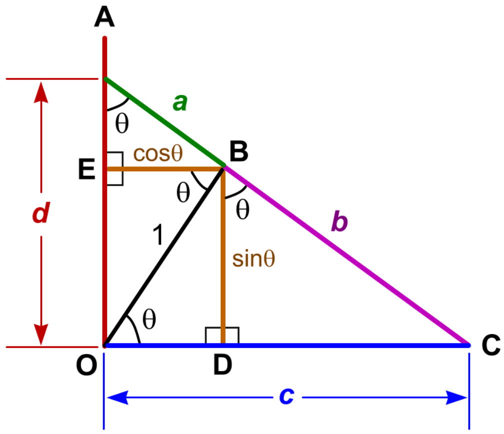
$$\frac{1}{\cos\theta} = \frac{b}{\sin\theta} \quad => \quad b = \frac{\sin\theta}{\cos\theta} = \tan\theta \quad => \quad b = \tan\theta$$

Next we look at $\Delta OBD \sim \Delta BCO$ to determine c. We should take the ratio of c to the adjacent side since I prefer not to mess with the b, which would work by the way since we know $b = \tan\theta$.

$$\frac{1}{\cos\theta} = \frac{c}{1} \quad => \quad c = \frac{1}{\cos\theta} \quad => \quad c = \sec\theta$$

But if you wanted to use the opposite b side in ΔBCO you would get the same result.

$$\frac{1}{\sin\theta} = \frac{c}{b} \quad => \quad c = \frac{b}{\sin\theta} \quad => \quad c = \frac{\tan\theta}{\sin\theta} \quad => \quad c = \frac{1}{\sin\theta}\frac{\sin\theta}{\cos\theta} \quad => \quad c = \frac{1}{\cos\theta} \quad => \quad c = \sec\theta$$



Finally, we want d. For a variety of approaches I note right off that

$$d = (a+b)\cos\theta .$$

I know $a = \cot\theta$ and $b = \tan\theta$.

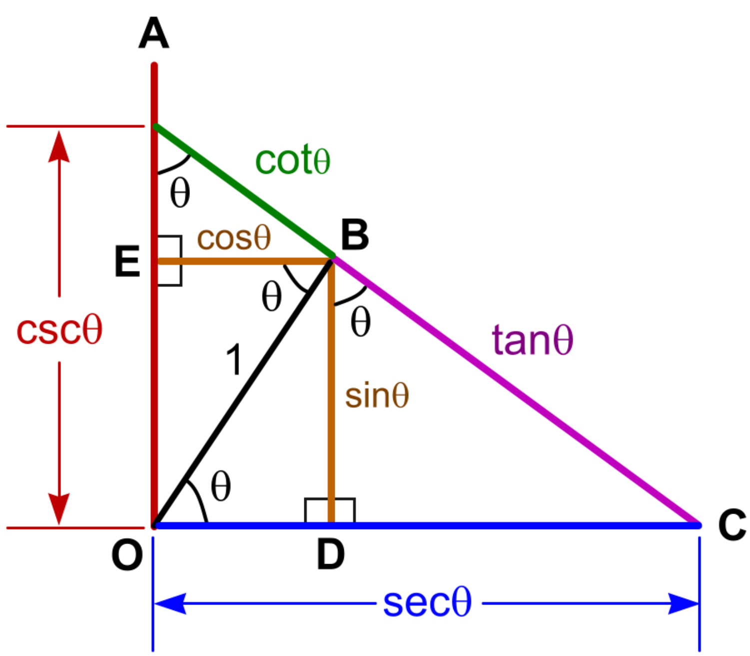
$$d = (a+b)\cos\theta \quad => \quad d = (\cot\theta + \tan\theta)\cos\theta$$

$$d = (\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta})\cos\theta \quad => \quad d = \frac{\cos^2\theta}{\sin\theta} + \sin\theta$$

$$=> \quad d = \frac{\cos^2\theta + \sin^2\theta}{\sin\theta} \quad => \quad d = \frac{1}{\sin\theta} \quad => \quad d = \csc\theta$$

In summary: $a = \cot\theta$, $b = \tan\theta$, $c = \sec\theta$, and $d = \csc\theta$.

We label these results in the figure below.



Bibliography

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