General Relativity
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Chapter GR-1 Notes. Vectors and Tensors

GR1-0. Introduction.

There are lots of fine texts on General Relativity and many resources on the Internet. You are encouraged to seek out such supplemental material. The notes you find here begin in a very basic way and our discussion is geared to undergraduates.

I developed notes c. 1980 for seniors (Joe Mitchell and Frank Keller) and a version with restricted topics for a freshman (Angelo Bencivenga). As we do not have a general relativity course at UNC Asheville, two sets of notes were developed specifically for students at different levels in their studies. Now, 35 years later, Dylan Cromer has made a similar request. Therefore, I am combining both sets of materials from the past and I am including more figures for this 2015-2016 edition. I am also expanding the notes to include overlapping topics.

Albert Einstein (1879-1955)
Schweizerische Landesbibliothek

Einstein photo from 1912, three years before arriving at General Relativity.

Einstein has always been an inspiration to physics students for over a century. The year 2015 marked the 100^{th} anniversary of the general theory of relativity – the year Einstein found his field equations.

I find the quote by Larry Smarr, a physicist and super computer expert, an excellent introduction to general relativity. His quote is:

"I think Einstein's Theory of Relativity is one of the most beautiful creations of human kind. It is both scientific and esthetic at the same time. It's one of the real moments in which beauty, in this case the mathematical beauty of equations, led to the science being discovered, and in its best form, science and art are indistinguishable." Larry Smarr, http://archive.ncsa.illinois.edu/Cyberia/NumRel/Smarr_2.html

Larry Smarr has used high performance computers to model distortions in spacetime by massive objects, including black holes.

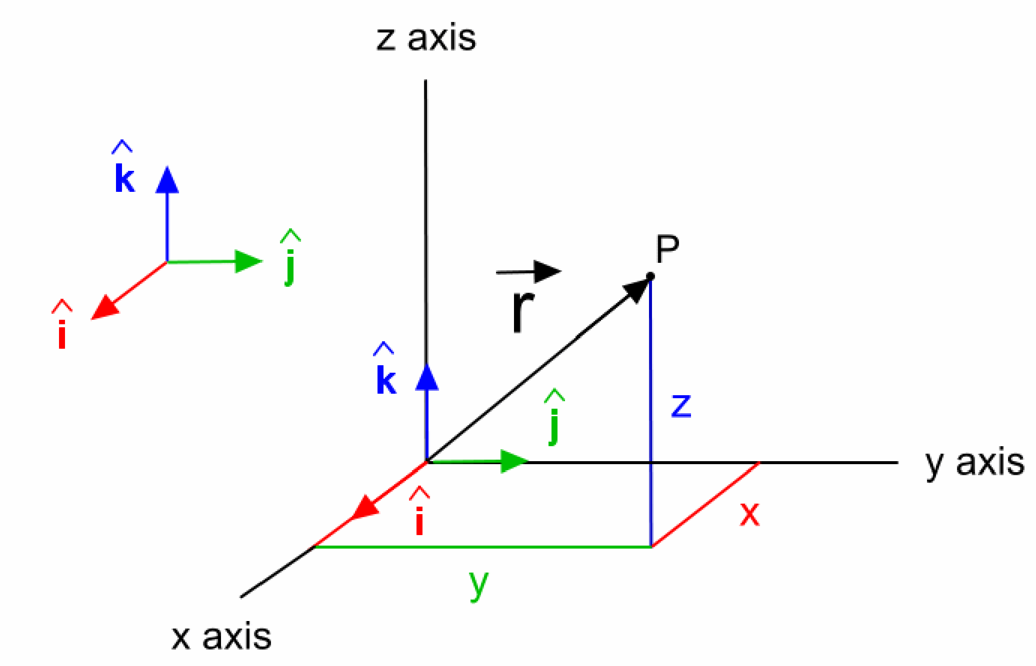
http://archive.ncsa.illinois.edu/Cyberia/NumRel/EinsteinEquations.html#expressed

We shall begin.

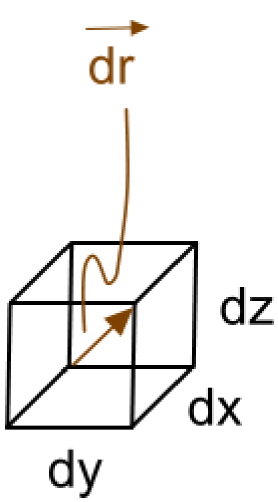
GR1-1. Contravariant Vectors.

The classic approach to general relativity abounds with vectors and tensors, which you will learn about. To start things off, an example of a vector is the position vector.

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$



The differential increment of the position vector is $\overrightarrow{dr} = dx\hat{i} + dy\hat{j} + dz\hat{k}$.

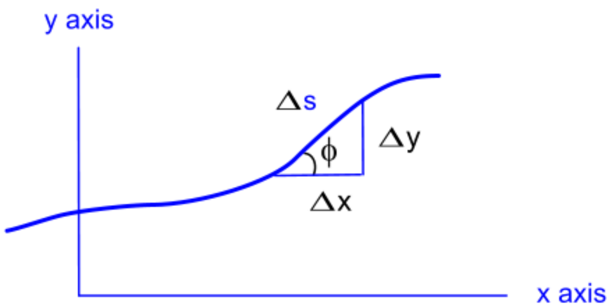


The square of the differential length, $ds^2 = dr^2$, is given by the Pythagorean Theorem since our unit vectors ($\hat{i}$, $\hat{j}$, and $\hat{k}$) are orthogonal.

$$ds^2 = dx^2 + dy^2 + dz^2$$

The concept of differential distance is extremely important in general relativity.

Here is a quick review of a derivative. The derivative is given by $\dfrac{dy}{dx} = \lim\limits_{x \to 0} \dfrac{\Delta y}{\Delta x}$,



which allows us to write $dy = \dfrac{dy}{dx} dx$.

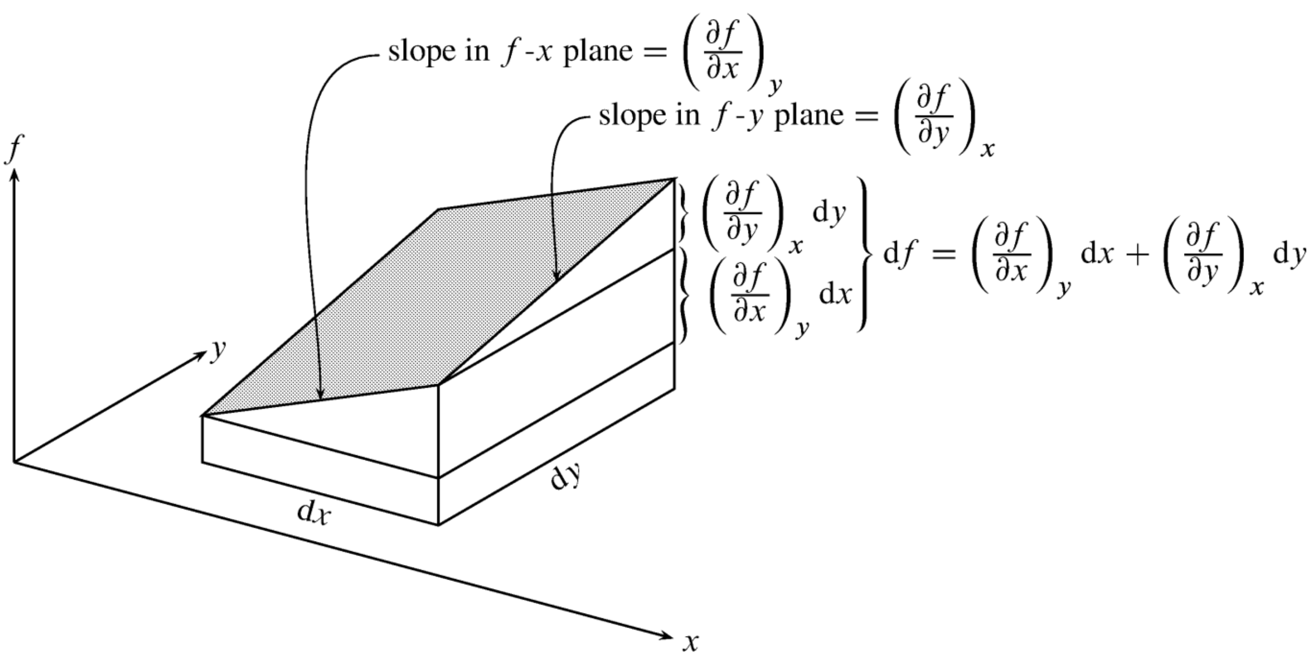
This equation states that the infinitesimal change in y is given by the derivative (of y with respect to x) times the infinitesimal change in x.

Watch what happens when we have a function f of two variables such that f = f(x,y).

The total change in f is then given by a sum where we use partial derivatives

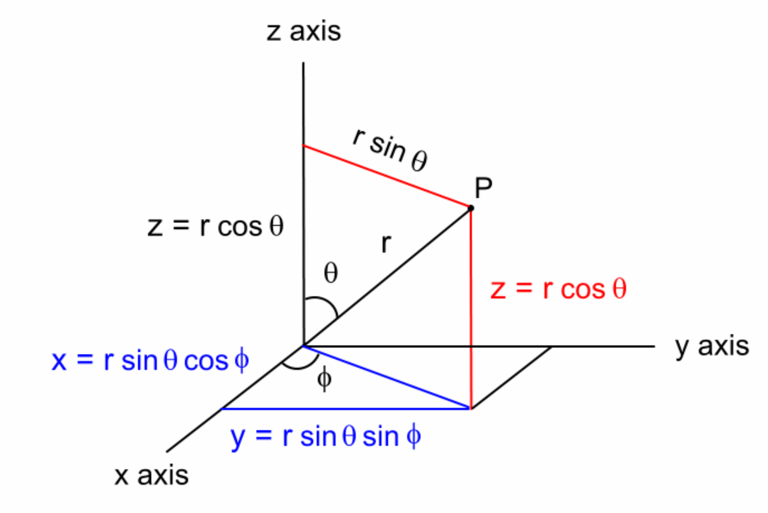
$$df = \left[\frac{\partial f}{\partial x}\right]_y dx + \left[\frac{\partial f}{\partial y}\right]_x dy = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy .$$

Subscripts mean we keep those variables constant, which is understood in the last version above. Refer to the following figure for a visualization of this important result.



Courtesy http://chemwiki.ucdavis.edu. Contributor Howard DeVoe, Associate Professor Emeritus, University of Maryland, from Thermodynamics and Chemistry.

The (x,y,z) coordinates are our Cartesian coordinates. We have other coordinate systems like cylindrical and spherical. In general, we can consider coordinates (u,v,w). The dictionary that lets us transform from one coordinate system to another consists of functions such as $u = u(x, y, z)$, $v = v(x, y, z)$, and $w = w(x, y, z)$.



As an example, the radial coordinate in spherical coordinates can be obtained from Cartesian coordinates by

$$r = \sqrt{x^2 + y^2 + z^2}\ .$$

For the other two spherical coordinates, we have

$$\phi = \tan^{-1}\frac{y}{x}$$ and

$$\theta = \cos^{-1}\frac{z}{\sqrt{x^2 + y^2 + z^2}}\ .$$

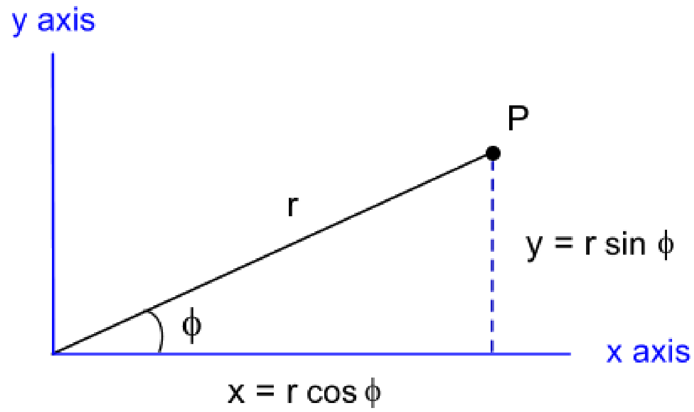
Getting back to $u = u(x, y, z)$, $v = v(x, y, z)$, and $w = w(x, y, z)$, from the laws of partial differentiation, we have

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz$$

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz\ .$$

To get some practice with partial derivatives, we consider two dimensions with Cartesian (x, y) and polar coordinates (r, ϕ).



We have the transformation equations

$$r = \sqrt{x^2 + y^2}$$

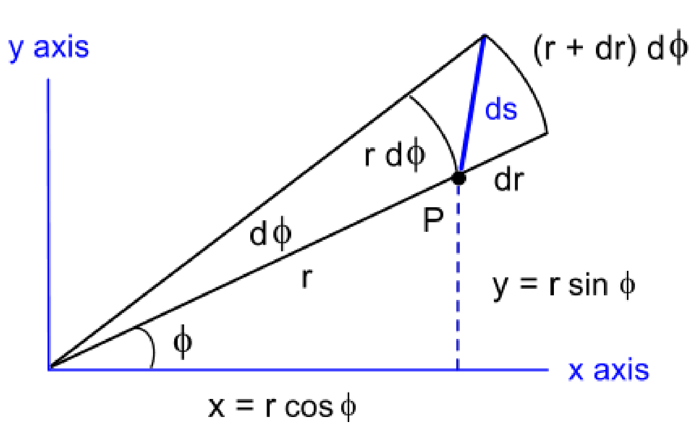
and

$$\phi = \tan^{-1}\frac{y}{x}.$$

Our former three dimensional equations now reduce to

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \quad \text{and} \quad dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

with $(u,v) = (r,\phi)$. You will be working with these in your first homework problem.



The square of the differential line element in polar coordinates is given from the figure as

$$ds^2 = dr^2 + [(r+dr)d\phi]^2$$

from the Pythagorean theorem. On expanding the second term, we can neglect the super small infinitesimal pieces $dr^2 d\phi^2$ and $2rdrd\phi^2$ as these contributions go to zero much faster than dr^2 and $d\phi^2$. Therefore, we arrive at

$$ds^2 = dr^2 + r^2 d\phi^2.$$

Homework HW-1. Polar Coordinates. Now it's time to practice. Calculate

$$dr = \frac{\partial r}{\partial x}dx + \frac{\partial r}{\partial y}dy$$

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy$$

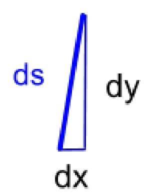
and demonstrate with detailed mathematical steps that

$$dr^2 + r^2 d\phi^2 = dx^2 + dy^2 \, .$$

Note that the square of the line element in Cartesian coordinates is our familiar

$$ds^2 = dx^2 + dy^2 \, .$$

See the same ds below for a visualization of the above result.



Let's define the Cartesian coordinates as $(x, y, z) \equiv (x^1, x^2, x^3)$ and the general coordinates (u, v, w) as (q^1, q^2, q^3). Then,

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz$$

$$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz$$

can be compactly written as

$$dq^i = \sum_{j=1}^{3} \frac{\partial q^i}{\partial x^j} dx^j \, .$$

The summation sign indicates that we should let the j-index go from 1 to 3 and add the three terms on the right. For the case where i = 1, the sum on j gives

$$dq^1 = \frac{\partial q^1}{\partial x^1} dx^1 + \frac{\partial q^1}{\partial x^2} dx^2 + \frac{\partial q^1}{\partial x^3} dx^3 \, ,$$

which is equivalent to

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz \, .$$

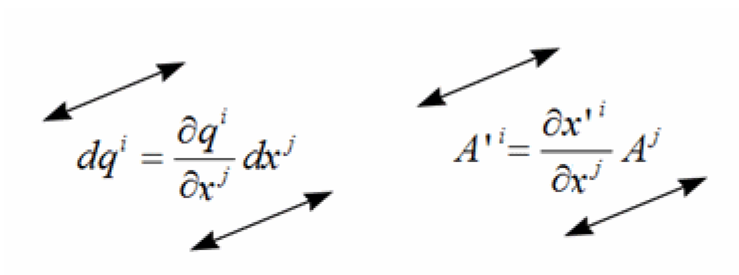
Since summation signs appear so frequently in general relativity, Einstein stopped writing them down by 1916. It is understood that we sum over an index appearing twice on the right side. This convention is called Einstein's summation convention. We can write

$$dq^i = \frac{\partial q^i}{\partial x^j} dx^j \quad \text{for} \quad dq^i = \sum_{i=j}^{3} \frac{\partial q^i}{\partial x^j} dx^j \, .$$

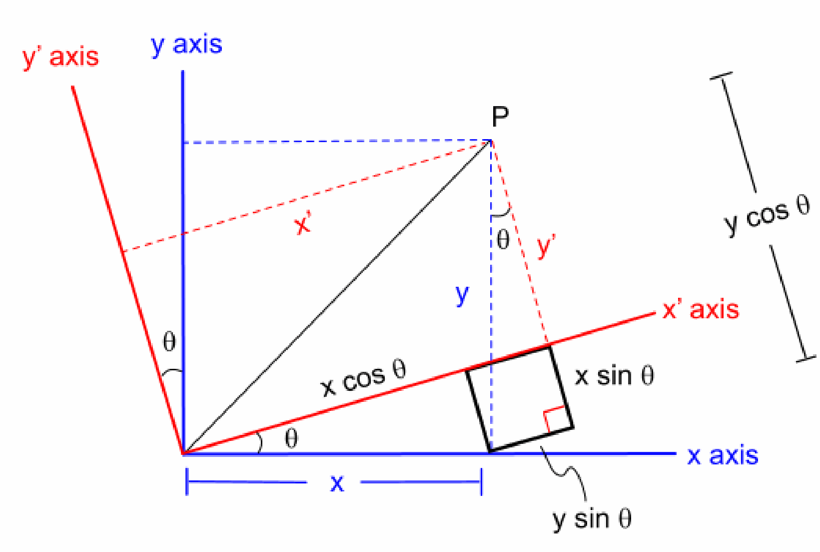
Vector components that transform like $dq^i = \dfrac{\partial q^i}{\partial x^j} dx^j$ are said to be contravariant. Consider a vector in two reference frames K and K' with vector $\vec{A}$ in the K frame and vector $\overrightarrow{A'}$ in the K' frame. The vector is contravariant if its components transform as

$$A'^i = \frac{\partial x'^i}{\partial x^j} A^j \, .$$

When you think contravariant, your primed variables and unprimed variables are situated along the slant as illustrated below. Note that the components for each frame of reference have the same index i or j. Remember that you sum on the double index j on the right side.



Homework HW-2. Rotational Transformation. Rotating a coordinate system provides for another example of a coordinate transformation. We will return to this transformation in more detail later.



$$x' = x\cos\theta + y\sin\theta$$

$$y' = y\cos\theta - x\sin\theta$$

$$dx' = \frac{\partial x'}{\partial x}dx + \frac{\partial x'}{\partial y}dy$$

$$dy' = \frac{\partial y'}{\partial x}dx + \frac{\partial y'}{\partial y}dy$$

Show that $ds'^2 \equiv dx'^2 + dy'^2 = dx^2 + dy^2 \equiv ds^2$.

GR1-2. Covariant Vectors.

You may have wondered about our placement of indices as superscripts. We defined general coordinates as (q^1, q^2, q^3). You have probably encountered subscripts for vector components in introductory physics, such as

$$\vec{A} = A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k} \ ,$$

rather than

$$\vec{A} = A^1 \hat{i} + A^2 \hat{j} + A^3 \hat{k} \ .$$

In a tensor course we have to be very careful here. Whether to use A^i or A_i matters. It depends how the components transform. In our last section we defined a contravariant vector as one whose components transform as below.

$$A'^i = \frac{\partial x'^i}{\partial x^j} A^j$$

Some vectors transform differently according to the transformation equations below.

$$A'_i = \frac{\partial x^j}{\partial x'^i} A_j$$

Such vectors are called covariant vectors. Note that you can write the contravariant transformation as

$$A'^i = \frac{\partial x'^i}{\partial x^j} A^j \quad \text{or} \quad A'^i = A^j \frac{\partial x'^i}{\partial x^j}$$

and the covariant one as

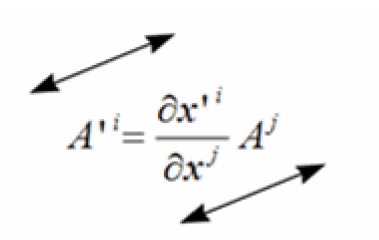
$$A'_i = \frac{\partial x^j}{\partial x'^i} A_j \quad \text{or} \quad A'_i = A_j \frac{\partial x^j}{\partial x'^i} \ .$$

The differentials dx^i transform in the contravariant manner as we have seen earlier:

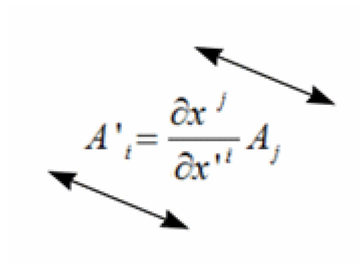
$$dx'^i = \frac{\partial x'^i}{\partial x^1} dx^1 + \frac{\partial x'^i}{\partial x^2} dx^2 + \frac{\partial x'^i}{\partial x^3} dx^3 = \frac{\partial x'^i}{\partial x^j} dx^j$$

Therefore, we write the differentials with superscripts in both cases below.

Contravariant Vector Transformation



Covariant Vector Transformation



An example of a covariant vector is the gradient. We take the gradient of the scalar function $f(x, y, z)$ below. This action promotes the scalar $f(x, y, z)$ to a vector quantity.

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

In a rotated coordinate system which we designate with primes,

$$\nabla f = \frac{\partial f}{\partial x'}\hat{i}' + \frac{\partial f}{\partial y'}\hat{j}' + \frac{\partial f}{\partial z'}\hat{k}' .$$

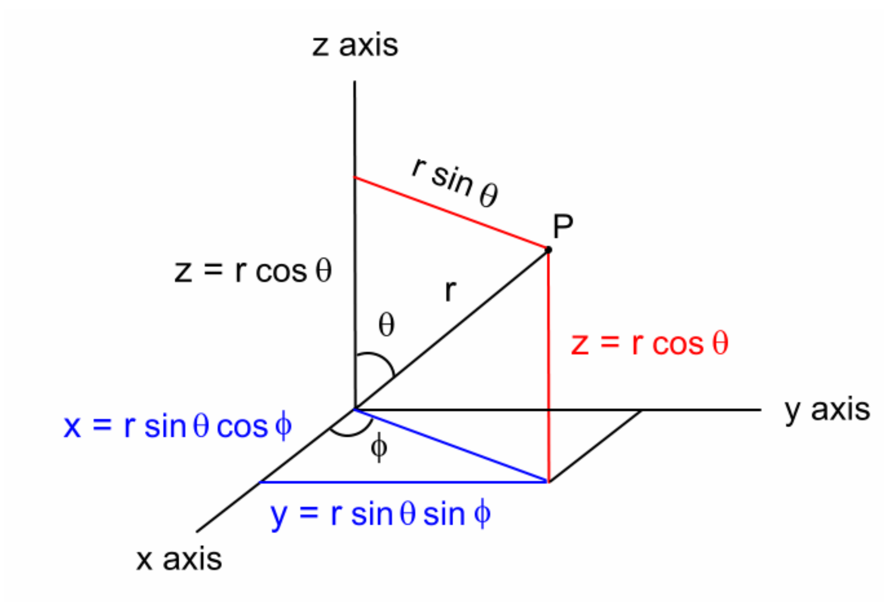
From the laws of partial derivatives, $\frac{\partial f}{\partial x'} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x'} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x'}$.

Let $A'_i = \frac{\partial f}{\partial x'^i}$ and $A_i = \frac{\partial f}{\partial x^i}$.

Then,

$$A'_i = A_j \frac{\partial x^j}{\partial x'^i} \quad \text{or} \quad A'_i = \frac{\partial x^j}{\partial x'^i} A_j .$$

Our next step is to apply the previous equations in the setting of polar coordinates. We use (r,ϕ) for polar coordinates so that θ is reserved for the spherical coordinate angle coming down from the z-axis. Note that the spherical coordinate r reduces to the polar coordinate r when $\theta = 90°$.



Let $\theta = 90°$ in the left figure. Then we are restricted to the x-y plane as z = 0.

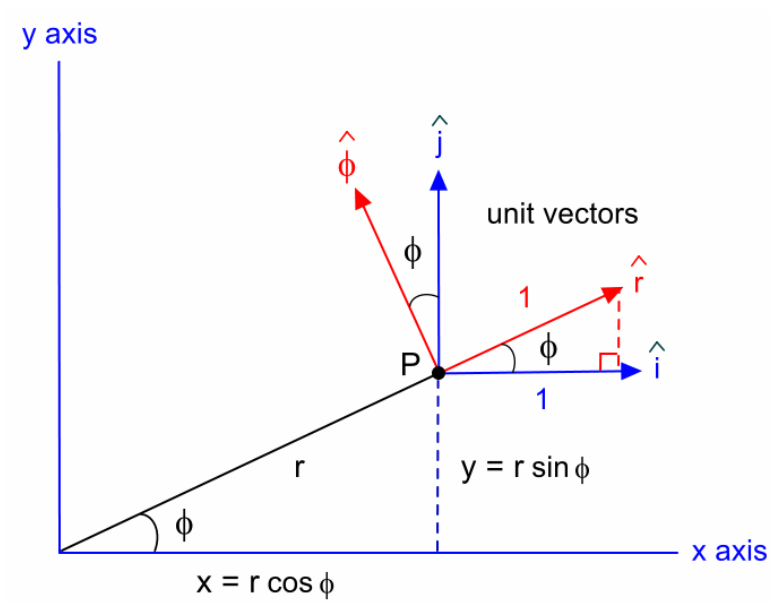
In the x-y plane, the coordinates reduce to the following.

$$x = r\cos\phi$$

$$y = r\sin\phi$$

$$z = 0$$

We will also need the transformation for the unit vectors. Convince yourself from the figure below that the following transformations hold.



$$x = r\cos\phi \quad y = r\sin\phi$$

$$r = \sqrt{x^2 + y^2}$$

$$\phi = \tan^{-1}\frac{y}{x}$$

$$\hat{r} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$$

$$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

Now solve the above two equations for $\hat{i}$ and $\hat{j}$ to arrive at

$$\hat{i} = \cos\phi\,\hat{r} - \sin\phi\,\hat{\phi} \quad \text{and} \quad \hat{j} = \sin\phi\,\hat{r} + \cos\phi\,\hat{\phi}.$$

We are ready for the homework problem.

Homework HW-3. The Gradient in Polar Coordinates. Consider two coordinate frames: 1) Cartesian coordinates (x, y) and 2) polar coordinates (r, ϕ).

Start with the gradient in Cartesian coordinates.

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

Now perform the covariant transformation of the form $A'_i = A_j \dfrac{\partial x^j}{\partial x'^i}$ to arrive at

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial y}.$$

Show that your result is

$$\nabla f = \left[\frac{\partial f}{\partial r}\cos\phi - \frac{\partial f}{\partial \phi}\frac{\sin\phi}{r}\right]\hat{i} + \left[\frac{\partial f}{\partial r}\sin\phi + \frac{\partial f}{\partial \phi}\frac{\cos\phi}{r}\right]\hat{j}.$$

To complete the transformation and get everything in polar coordinates, we need to substitute in

$$\hat{i} = \cos\phi\hat{r} - \sin\phi\hat{\phi} \quad \text{and} \quad \hat{j} = \sin\phi\hat{r} + \cos\phi\hat{\phi}.$$

Do this substitution and show that

$$\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{\phi}.$$

From the above result, the gradient in polar coordinates can be written as

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial}{\partial \phi},$$

where we are careful to put the unit vectors first since they are not constants.

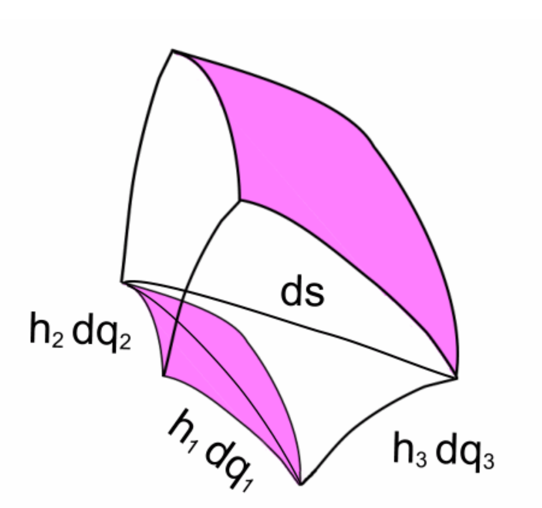
You have essentially, by the way, also derived the gradient in cylindrical coordinates since the z-axis is a Cartesian one. In three dimensions, we will use ρ as the cylindrical radial coordinate so as to never be confused with the spherical radial coordinate r . Then, the gradient in cylindrical coordinates is

$$\nabla = \rho \frac{\partial}{\partial \rho} + \hat{\phi} \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{k} \frac{\partial}{\partial z} .$$

GR1-3. Curvilinear Orthogonal Coordinates.

From our study of the gradient, you realize that it can take some mathematical manipulation to arrive at the gradient in a non-Cartesian coordinate system. We would like a shortcut in finding gradients and other things in general orthogonal coordinate systems. Note that even though polar coordinates have unit vectors that depend on location, the unit vectors are still perpendicular to each other, i.e., orthogonal. This also goes for spherical coordinates. **We are going to use subscripts for differentials here.**

Let’s consider an infinitesimal solid in general curvilinear orthogonal coordinates. Note that each differential variable has a factor in front. These "h" factors are called scale factors. A familiar example is the arc length in polar coordinates: $ds = r d\theta$. The r is a scale factor.



Differential Line Element. We use the Pythagorean theorem twice for the diagonal of the solid. First, for the base, we have for the square of the hypotenuse

$$h_1^2 dq_1^2 + h_2^2 dq_2^2 .$$

Second, adding to this the square of the height gives

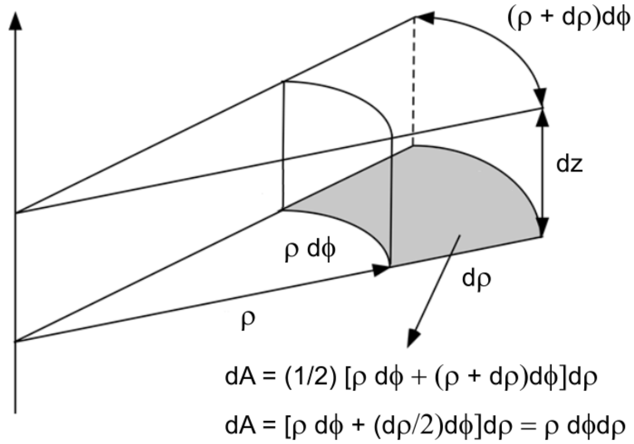
$$ds^2 = h_1^2 dq_1^2 + h_2^2 dq_2^2 + h_3^2 dq_3^2 .$$

Differential Volume Element. We can use the “length times width times height” idea to arrive at the volume element.

$$dV = (h_1 dq_1)(h_2 dq_2)(h_3 dq_3) = h_1 h_2 h_3 dq_1 dq_2 dq_3$$

Let’s check these out with cylindrical coordinates. We first work things out the long way.

Cylindrical Coordinates Differential Line and Volume Elements. Figure from Tony Saad.



For the volume element we write $dV = dAdz$, where the dA is the area of the base in the figure. This is simply $dA = \rho d\phi d\rho$. But if you want to be super careful, use the average of the shorter and longer arc lengths (see left). As we take the limit of the infinitesimal, a product of three differentials will vanish faster than the product of two. Therefore, we can toss the $d\rho d\phi d\rho$ term compared to the $d\phi d\rho$ term. The volume element is then

$$dV = \rho d\phi d\rho dz \,.$$

The line element for the diagonal of the "cube-like" region is found by using the Pythagorean theorem twice: once for the floor $d\rho^2 + \rho^2 d\phi^2$ and then the diagonal of the floor with the height dz.

$$ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$$

From $d\rho$, $\rho d\phi$, and dz in the figure above, we identify for cylindrical coordinates:

$$q_1 = \rho,\ q_2 = \phi,\ q_3 = z \text{ with } h_1 = 1,\ h_2 = \rho,\ h_3 = 1.$$

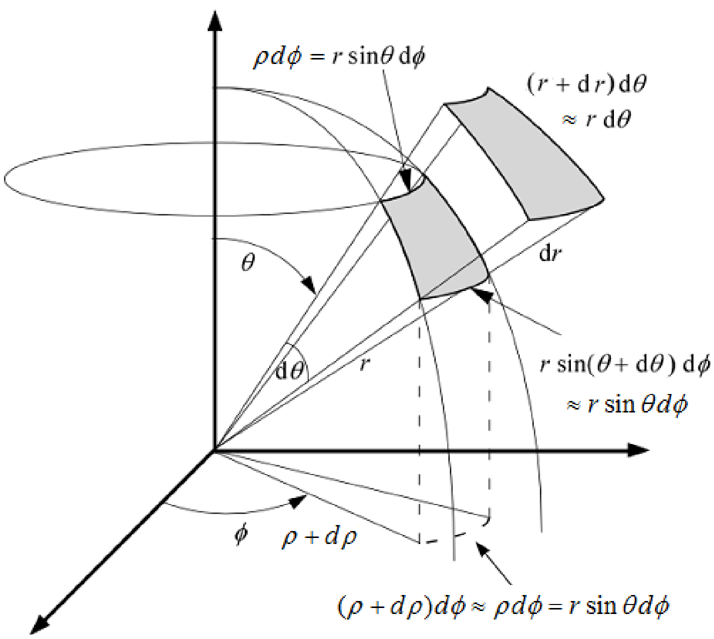
Using the coordinates and scale factors for cylindrical coordinates we can find dV and ds^2 from the master formulas

$$ds^2 = h_1^2 dq_1^2 + h_2^2 dq_2^2 + h_3^2 dq_3^2 \quad \text{and} \quad dV = (h_1 dq_1)(h_2 dq_2)(h_3 dq_3) \,.$$

With the values for the above h scale factors:

$$ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2 \quad \text{and} \quad dV = \rho d\phi d\rho dz \,.$$

Spherical Coordinates Differential Line and Volume Elements. Figure from Tony Saad.



Let's do the same for spherical coordinates. The volume element is found using length times width times height for the tilted volume element.

$$dV = (r\sin\theta d\phi)(rd\theta)dr$$

We don't even take the average of the widths since to first order, the differential

$$\left[r\sin(\theta + d\theta)\right]d\phi$$

becomes $r\sin\theta d\phi$. The volume element is usually written as

$$dV = r^2 \sin\theta dr d\theta d\phi.$$

The differential line element is the square of the diagonal as before. So we use simple square each of the differential sides. $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$

From dr, $rd\theta$, and $r\sin\theta d\phi$ in the figure above, we identify for spherical coordinates:

$$q_1 = r,\ q_2 = \theta,\ q_3 = \phi \text{ with } h_1 = 1,\ h_2 = r,\ h_3 = r\sin\theta.$$

Using the coordinates and scale factors for spherical coordinates we can confirm dV and ds^2 from the master formulas

$$ds^2 = h_1^2 dq_1^2 + h_2^2 dq_2^2 + h_3^2 dq_3^2 \qquad dV = (h_1 dq_1)(h_2 dq_2)(h_3 dq_3)$$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \qquad dV = r^2 \sin\theta dr d\theta d\phi$$

As we have seen, the Gradient in Cartesian coordinates is given by

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\ .$$

We can obtain the gradient in curvilinear coordinates by comparing the differentials. In Cartesian coordinates we have these three differentials:

dx, dy, and dz.

In cylindrical coordinates we have: $d\rho$, $\rho d\phi$, and dz. Note that each has dimension of length. Our gradient in cylindrical coordinates is then

$$\nabla f = \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{k}\ .$$

This is the shortcut. We just write the answer down since we know about scale factors.

In spherical coordinates we have: dr, $rd\theta$, and $r\sin\theta d\phi$. Note that each has dimension of length. Our gradient in spherical coordinates is then

$$\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}\ .$$

In curvilinear coordinates we have: $h_1 dq_1$, $h_2 dq_2$, and $h_3 dq_3$. Note that each has dimension of length. Our gradient in curvilinear coordinates is then

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial q_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial q_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial q_3}\hat{e}_3\ .$$

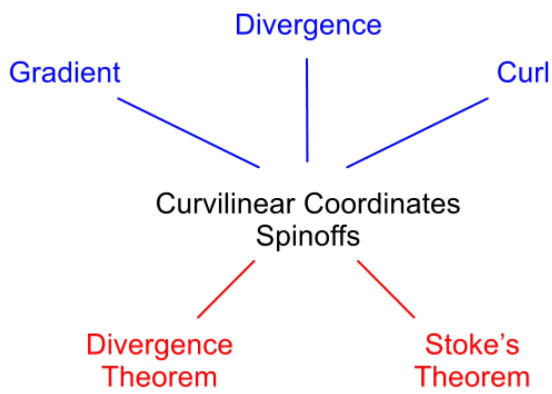
Caution: If you write the operator by itself, put the unit vectors on the left since they are in general functions of the coordinates.

$$\nabla = \hat{e}_1\frac{1}{h_1}\frac{\partial}{\partial q_1} + \hat{e}_2\frac{1}{h_2}\frac{\partial}{\partial q_2} + \hat{e}_3\frac{1}{h_3}\frac{\partial}{\partial q_3}$$

GR1-4. Advanced Vector Analysis – Curvilinear Coordinates.

We will take a detour here to use the curvilinear concept to derive three more very important operators in engineering and physics: divergence, curl, and Laplacian. But with the power of curvilinear coordinates, we can arrive at the very important above operators in a relatively short time. One of the strengths of studying General Relativity is the spinoff effect – you learn on the side very important concepts used throughout engineering and physics. In this section we introduce some basic definitions first.

The spinoffs in this chapter will give you tremendous strength in theoretical physics. Besides the operators, we will also see the general forms for the Divergence Theorem and Stoke's Theorem, again extremely important in engineering and physics.



The gradient, divergence, and curl can be defined with the del operator

$$\nabla \equiv \frac{\partial}{dx}\hat{i} + \frac{\partial}{dy}\hat{j} + \frac{\partial}{dz}\hat{k} .$$

The gradient of a scalar f gives us a vector: $\nabla f = \frac{\partial f}{dx}\hat{i} + \frac{\partial f}{dy}\hat{j} + \frac{\partial f}{dz}\hat{k}$

Homework HW-4. Divergence and Curl. If you have any trouble with this problem, go immediately to the Appendix Review at the end of this chapter.

Show that the divergence of a vector $\vec{E} = E_x\hat{i} + E_y\hat{j} + E_z\hat{k}$ gives a scalar:

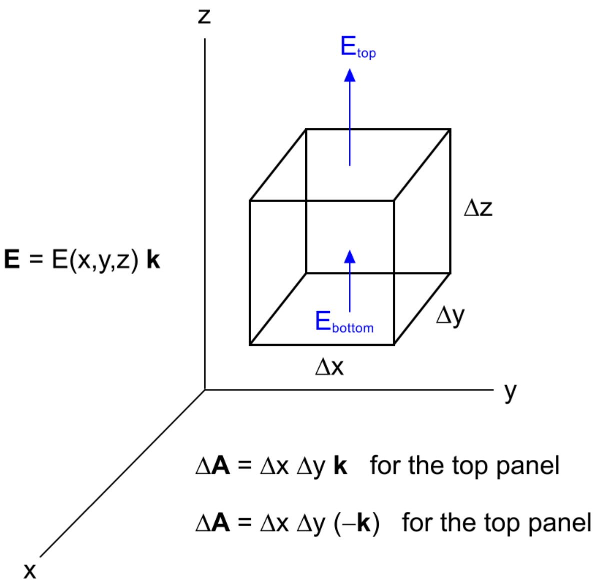
$\nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$ from your definition of the dot product with unit vectors.

Show that the curl of a vector $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$ gives a vector:

$$\nabla \times \vec{B} = \hat{i}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) + \hat{j}\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) + \hat{k}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right),$$

from you definitions of the cross product with unit vectors.

GR1-5. The Divergence Theorem. Here is a simplified derivation of the divergence theorem in Cartesian coordinates.



We are interested in calculating the flux through the enclosed surface, which we write as

$$\oiint \vec{E} \cdot \vec{dA}.$$

$E_{bottom} = E_z(x, y, z)$ and its counterpart $E_{top} = E_z(x, y, z + \Delta z)$.

The net flux out of the surface of our cube is given by multiplying the magnitude of the perpendicular vector component that pierces each surface. Here we have top and bottom. We subtract what goes in from what goes out.

$$\oiint \vec{E} \cdot \vec{dA} => E_z(x, y, z + \Delta z)\Delta x \Delta y - E_z(x, y, z)\Delta x \Delta y$$

$$\oiint \vec{E} \cdot \vec{dA} => \frac{E_z(x, y, z + \Delta z) - E_z(x, y, z)}{\Delta z} \Delta x \Delta y \Delta z$$

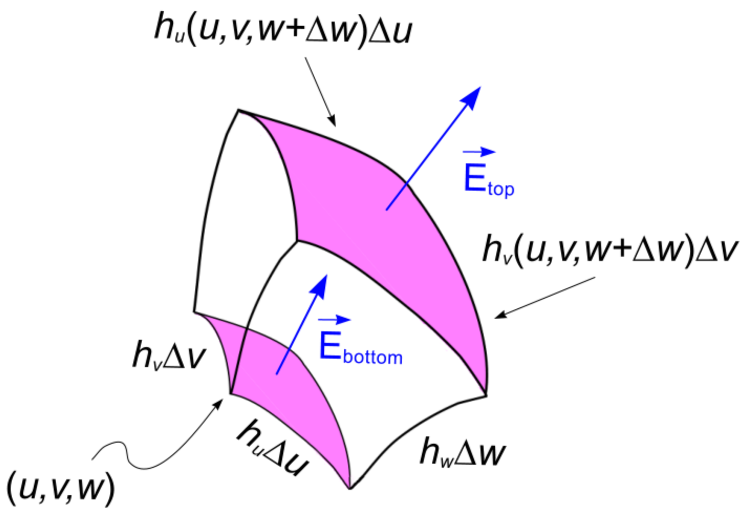
$$\oiint \vec{E} \cdot \vec{dA} = \iiint \frac{\partial E_z}{\partial z} dxdydz$$ Note left surface integral and right volume integral.

$$\oiint \vec{E} \cdot \vec{dA} = \iiint \left[\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right] dxdydz$$

The Divergence Theorem: $$\oiint \vec{E} \cdot \vec{dA} = \iiint \left[\nabla \cdot \vec{E} \right] dxdydz$$

Homework HW-5. Show that starting with $E_{bottom} = E_z(x + \frac{\Delta x}{2}, y + \frac{\Delta y}{2}, z)$ and its appropriate counterpart that you get the same result. You will find that taking limits as the extra deltas go to zero do not lead to any derivatives. They just go away.

Now we are going to derive the divergence theorem in general for curvilinear orthogonal coordinates. We consider $(q_1, q_2, q_3) = (u, v, w)$ with scale factors (h_u, h_v, h_w). It is easier to work without subscripts for the coordinates.



Remember that the h scale factors can be functions of the coordinates

$$h_w = h_w(u, v, w).$$

We will take a simplified vector field such that

$$\vec{E}(u, v, w) = E_w \hat{e}_w.$$

Flux is defined as the product of the magnitude of the vector component piercing an area times that area. The net flux is what goes out minus what comes in. The only relevant areas for the volume element are the top and bottom.

$$\vec{E}_{bottom} = E_w(u, v, w)\hat{e}_w$$

$$\vec{E}_{top} = E_w(u, v, w + \Delta w)\hat{e}_w$$

$$dA_{bottom} = \left[h_u(u, v, w)\Delta u\right]\left[h_v(u, v, w)\Delta v\right]$$

$$dA_{top} = \left[h_u(u, v, w + \Delta w)\Delta u\right]\left[h_v(u, v, w + \Delta w)\Delta v\right]$$

Therefore, $\oiint \vec{E} \cdot \vec{dA} =>$

$$E_w(u, v, w + \Delta w)\left[h_u(u, v, w + \Delta w)\Delta u\right]\left[h_v(u, v, w + \Delta w)\Delta v\right]$$
$$-E_w(u, v, w)\left[h_u(u, v, w)\Delta u\right]\left[h_v(u, v, w)\Delta v\right]$$

$$\oiint \vec{E} \cdot \vec{dA} => \left[(E_w h_u h_v)\big|_{w+\Delta w} - (E_w h_u h_v)\big|_{w}\right]\Delta u \Delta v$$

$$\oiint \vec{E}\cdot\overrightarrow{dA} => \left[\frac{(E_w h_u h_v)\big|_{w+\Delta w} - (E_w h_u h_v)\big|_w}{\Delta w}\right]\Delta u \Delta v \Delta w$$

$$\oiint \vec{E}\cdot\overrightarrow{dA} => \frac{\partial(E_w h_u h_v)}{\partial w}\Delta u \Delta v \Delta w$$

Now we want the differential volume element

$$dV = (h_u\Delta u)(h_v\Delta v)(h_w\Delta w) = h_u h_v h_w \Delta u \Delta v \Delta w$$ in there.

$$\oiint \vec{E}\cdot\overrightarrow{dA} => \frac{1}{h_u h_v h_w}\frac{\partial(E_w h_u h_v)}{\partial w} h_u h_v h_w \Delta u \Delta v \Delta w$$

Now it's time to put the q-variables back in.

$$\oiint \vec{E}\cdot\overrightarrow{dA} => \frac{1}{h_1 h_2 h_3}\frac{\partial(E_3 h_1 h_2)}{\partial w} h_1 h_2 h_3 \Delta q_1 \Delta q_2 \Delta q_3$$

$$\oiint \vec{E}\cdot\overrightarrow{dA} = \iiint \frac{1}{h_1 h_2 h_3}\frac{\partial(E_3 h_1 h_2)}{\partial q_3} dV$$

From this we can generalize to the general form of the divergence.

$$\nabla\cdot\vec{E} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(E_1 h_2 h_3)}{\partial q_1} + \frac{\partial(E_2 h_1 h_3)}{\partial q_2} + \frac{\partial(E_3 h_1 h_2)}{\partial q_3}\right]$$

Homework HW-6. Show that the divergence in curvilinear coordinates reduces to the following in Cartesian, cylindrical, and spherical coordinates.

Cartesian: $$\nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

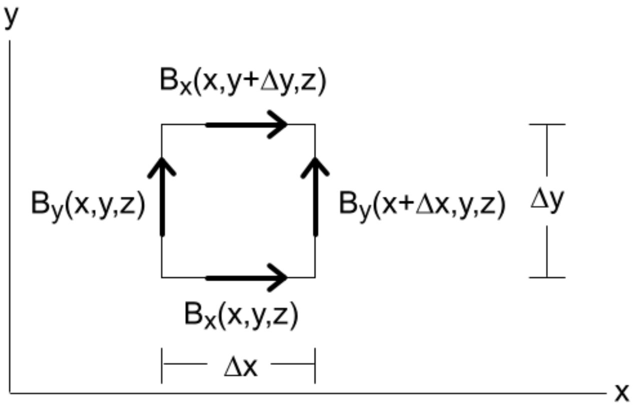
Cylindrical: $$\nabla\cdot\vec{A} = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

Spherical: $$\nabla\cdot\vec{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

GR1-6. Stoke's Theorem. Here we consider a vector field **B** and proceed to do a closed line integral of this field.

$$\oint \vec{B}\cdot\vec{dl}\ .$$

To simplify, we will pick the field to be in the x-y plane.



We obtain $\oint \vec{B}\cdot\vec{dl} \Rightarrow$

$$B_x(x,y,z)\Delta x + B_y(x+\Delta x,y,z)\Delta y - B_x(x,y+\Delta y,z)\Delta x - B_y(x,y,z)\Delta y\text{, i.e.,}$$

$$\frac{B_y(x+\Delta x,y,z)-B_y(x,y,z)}{\Delta x}\Delta x\Delta y - \frac{B_x(x,y+\Delta y,z)-B_x(x,y,z)}{\Delta y}\Delta x\Delta y\ .$$

Homework HW-7. Complete the derivation of Stoke's Theorem.

$$\oint \vec{B}\cdot\vec{dl} = \iint_A (\nabla\times\vec{B})\cdot\vec{dA}$$ Note left line integral and right surface integral.

We can extend this to curvilinear coordinates (u,v,w) with scale factors (h_u,h_v,h_w).

$$\oint \vec{B}\cdot\vec{dl} \Rightarrow \frac{\left[B_v(u,v,w)h_v(u,v,w)\right]_{u+\Delta u}-\left[B_v(u,v,w)h_v(u,v,w)\right]_u}{\Delta u}\Delta u\Delta v$$

$$-\frac{\left[B_u(u,v,w)h_u(u,v,w)\right]_{v+\Delta v}-\left[B_u(u,v,w)h_u(u,v,w)\right]_v}{\Delta v}\Delta u\Delta v$$

$$\oint \vec{B}\cdot\vec{dl} = \frac{1}{h_u h_v}\iint_A \left[\frac{\partial(B_v h_v)}{\partial u} - \frac{\partial(B_u h_u)}{\partial v}\right]dA_w$$

Introducing the q-variables $(q_1, q_2, q_3) = (u, v, w)$ with (h_1, h_2, h_3).

$$\oint \vec{B} \cdot \vec{dl} = \frac{1}{h_1 h_2} \iint_A \left[\frac{\partial (B_2 h_2)}{\partial q_1} - \frac{\partial (B_1 h_1)}{\partial q_2} \right] dA_3$$

The third component of the cross product defined as

$$(\nabla \times \vec{B})_3 = \frac{1}{h_1 h_2} \left[\frac{\partial (B_2 h_2)}{\partial q_1} - \frac{\partial (B_1 h_1)}{\partial q_2} \right].$$

leads to Stoke's Theorem in curvilinear coordinates.

$$\oint \vec{B} \cdot \vec{dl} = \iint_A (\nabla \times \vec{B}) \cdot \vec{dA}$$

Our cross product $\nabla \times \vec{B} = \frac{1}{h_1 h_2} \left[\frac{\partial (B_2 h_2)}{\partial q_1} - \frac{\partial (B_1 h_1)}{\partial q_2} \right] \hat{e}_3$ is better written as

$$\nabla \times \vec{B} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial (B_2 h_2)}{\partial q_1} - \frac{\partial (B_1 h_1)}{\partial q_2} \right] h_3 \hat{e}_3$$

Then we can use the following determinant to express this.

$$\nabla \times \vec{B} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \dfrac{\partial}{\partial q_1} & \dfrac{\partial}{\partial q_2} & \dfrac{\partial}{\partial q_3} \\ h_1 B_1 & h_2 B_2 & h_3 B_3 \end{vmatrix}$$

Conceptual Summary: The Divergence Theorem equates an enclosed vector-piercing surface integral with the divergence of the vector integrated over the enclosed volume. Stoke's Theorem equates a vector-projected-on-a-line loop integral with the perpendicular component of the curl of that vector integrated over the surface enclosed by the loop.

GR1-7. The Laplacian. We find the Laplacian of a function, i.e., $\nabla^2 f$, by applying the divergence to the gradient of a function: $\nabla^2 f = \nabla \cdot (\nabla f)$. We start with our previous result for the gradient

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial q_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial q_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial q_3}\hat{e}_3 .$$

and then use our previous result for the divergence

$$\nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(A_1 h_2 h_3)}{\partial q_1} + \frac{\partial(A_2 h_1 h_3)}{\partial q_2} + \frac{\partial(A_3 h_1 h_2)}{\partial q_3}\right],$$

where $\vec{A} = \nabla f$. So we substitute

$$A_1 = \frac{1}{h_1}\frac{\partial f}{\partial q_1},\ A_2 = \frac{1}{h_2}\frac{\partial f}{\partial q_2},\ \text{and}\ A_3 = \frac{1}{h_3}\frac{\partial f}{\partial q_3}.$$

We obtain for $\nabla^2 f$ the following.

$$\frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{1}{h_1}\frac{\partial f}{\partial q_1}h_2 h_3\right) + \frac{\partial}{\partial q_2}\left(\frac{1}{h_2}\frac{\partial f}{\partial q_2}h_1 h_3\right) + \frac{\partial}{\partial q_3}\left(\frac{1}{h_3}\frac{\partial f}{\partial q_3}h_1 h_2\right)\right]$$

This simplifies to

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial f}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial q_3}\right)\right].$$

Homework HW-8. Show that our Laplacian in curvilinear coordinates, which is

$$\nabla^2 = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(\frac{h_2 h_3}{h_1}\frac{\partial}{\partial q_1}) + \frac{\partial}{\partial q_2}(\frac{h_1 h_3}{h_2}\frac{\partial}{\partial q_2}) + \frac{\partial}{\partial q_3}(\frac{h_1 h_2}{h_3}\frac{\partial}{\partial q_3})\right],$$

reduces to the following in Cartesian, cylindrical, and spherical coordinates.

Cartesian: $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

Cylindrical: $\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho \frac{\partial}{\partial \rho}) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$

Spherical: $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial}{\partial r}) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta \frac{\partial}{\partial \theta}) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}$

SAMUEL S. ENSOR

Many years ago Mr. Samuel S. Ensor, my Calculus teacher at St. Joseph's College (SJ) in Philadelphia (now University) gave us a project in Calculus III that was long, but very useful and productive (Spring 1969). It is given below. Everyone aspiring to be a physicist or engineer should do this calculation once sometime in their studies. It will correct any rough edges you have in taking partial derivatives and using the chain rule.

Recommended Problem. Derive the Laplacian in spherical coordinates the long way! Start with

$$x = r\sin\theta\cos\phi\,,\ \ y = r\sin\theta\sin\phi\,,\ \ z = r\cos\theta$$

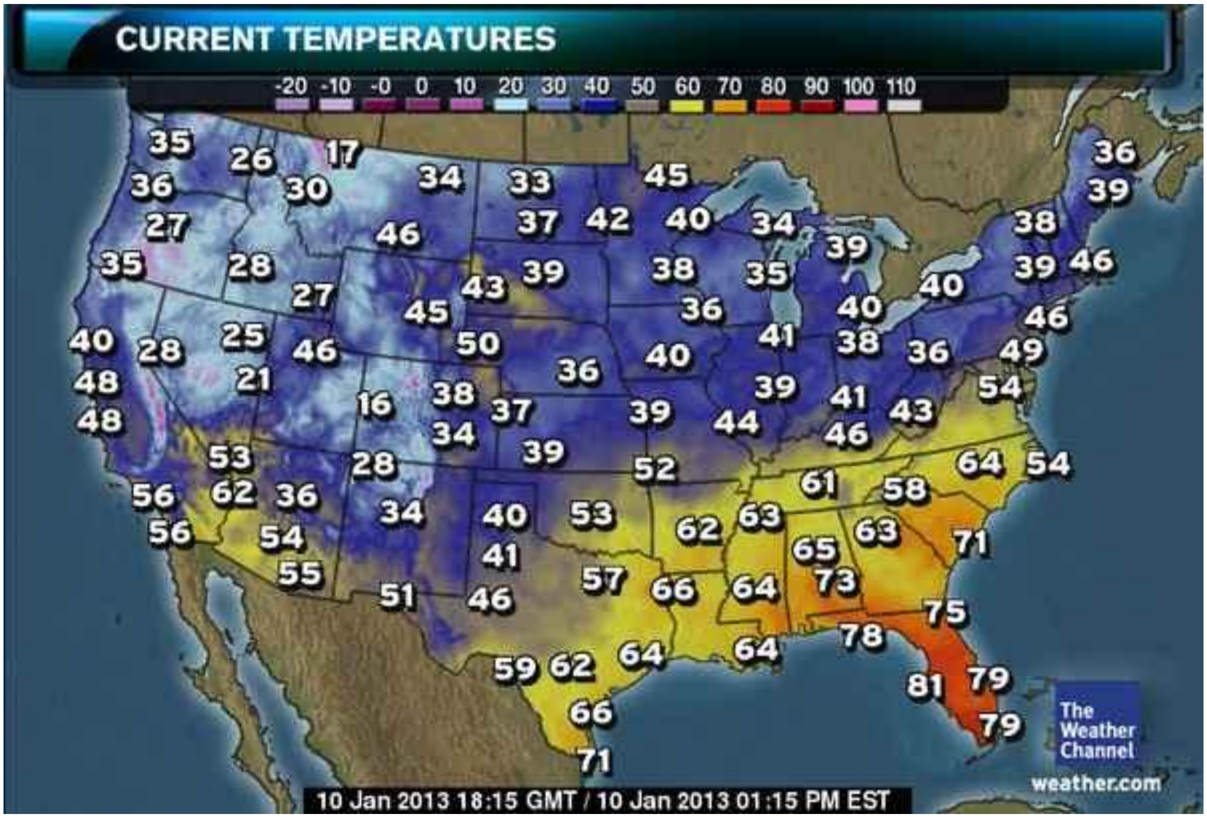
and you want $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

So you start cranking: $\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi}$ and so on. Have fun!

Photo Courtesy Ancestry.com, scan from the *Greatonion*, SJ 1953 Yearbook.

GR1-8. Tensors.

Tensor of Rank 0. The Rank 0 case is your scalar. A single number is all you need. There is no directional vector or anything like that. The length of a vector stripped of its direction is a scalar. Another example is temperature at each point in a room: T = T(x,y,z) or you can add the time variable so the temperatures change in time. Below is a snapshot of the temperatures across the United States.



Courtesy The Weather Channel

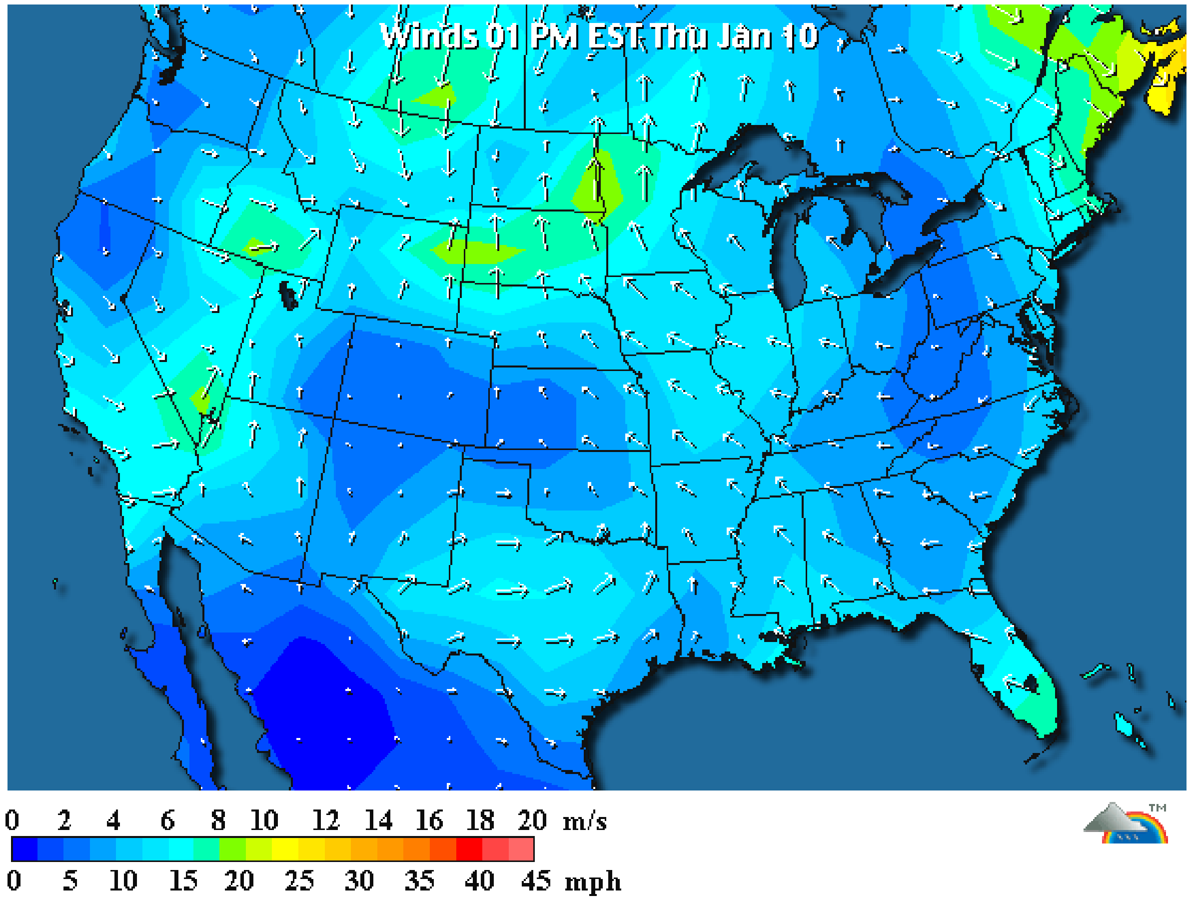
Tensor of Rank 1. This case is your vector. It has magnitude and direction. It can also be a function of the spatial coordinates as well as time. We have already seen the basic classifications of vectors according to their transformation properties.

Contravariant Vector A^i

$$A'^i = \frac{\partial x'^i}{\partial x^j} A^j$$

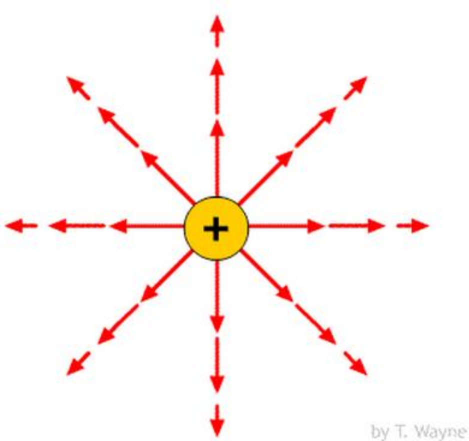
Covariant Vector A_i

$$A'_i = \frac{\partial x^j}{\partial x'^i} A_j$$



Courtesy Weather Underground, Inc.

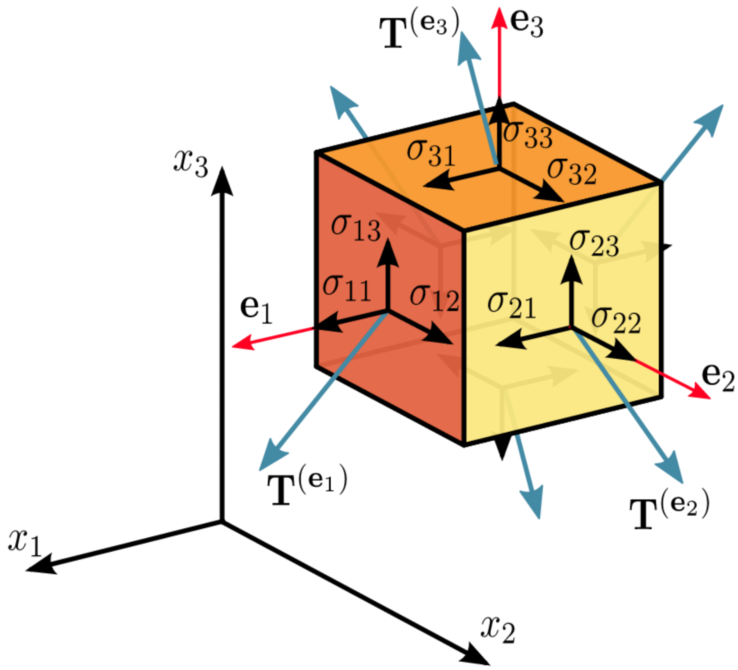
Wind velocity has magnitude (the speed) and direction. The length of the vector arrows indicate the magnitude of the velocity and the arrow points in the direction of the wind. Technically, speed is a scalar, the magnitude. When you promote speed to a vector you add the direction. However, often velocity is used informally for just speed.



Charge Image Courtesy Tony Wayne

Here is a vector field produced by a plus charge. Note the symmetry as all vectors points outward away from the positive charge. Also note that the lengths of the vectors decrease as you get farther away from the charge. The strength weakens according to the inverse square law. In contrast to the weather case, this field has a simple formula.

Tensor of Rank 2. Among friends, you can think of a tensor of rank 2 as needing 3 x 3 = 9 components, in three-dimensional space.



Courtesy Sanpaz, Wikipedia

The stress tensor is an example. We need to consider the force on each of the three main faces defined by the three unit vectors. On each surface there is a normal force and two shear (sideway) forces.

We need not consider all 6 faces since mechanical equilibrium guarantees that there will be opposing forces and torques on the opposite sides.

We need 9 quantities to define the stress.

Matrix notation will assist us here. For the tensors of Rank 0, 1, and 2 respectively, we can write for three dimensional space.

$$s = [T] = T \qquad \vec{A} = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \qquad \sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

For two dimensions we have for tensors of Rank 0, 1, and 2 respectively listed below.

$$s = [T] = T \qquad \vec{A} = \begin{bmatrix} A_x \\ A_y \end{bmatrix} \qquad M_{ij} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

But not all matrices are tensors. There are transformation properties that need to be satisfied. However, if you need vector components as in the stress analysis, then you are on good grounds that you are probably dealing with a tensor.

The transformation properties of tensors involved partial derivatives.

We know that a contravariant vector A^i transforms as

$$A'^i = \frac{\partial x'^i}{dx^j} A^j \ .$$

This vector is a contravariant tensor of Rank 1. A contravariant tensor of Rank 2 transforms with two partial derivatives in place.

$$T^{ij}{}' = \frac{\partial x^i{}'}{\partial x^m} \frac{\partial x'^j}{\partial x^n} T^{mn}$$

Similarly, a covariant vector transforms as

$$A'_i = \frac{\partial x^j}{\partial x'^i} A_j$$

and a covariant tensor of Rank 2 transforms like

$$T'_{ij} = \frac{\partial x^m}{\partial x'^i} \frac{\partial x^n}{\partial x'^j} T_{mn} \, .$$

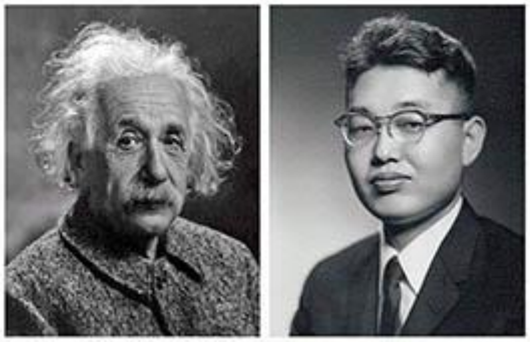
You can also have mixed types. Below is the transformation property for a tensor of Rank 3 with two covariant indices and one contravariant index.

$$S'_{ij}{}^k = \frac{\partial x^m}{\partial x'^i} \frac{\partial x^n}{\partial x'^j} \frac{\partial x'^k}{\partial x^p} S_{mn}{}^p$$

Homework HW-9. Give the transformation for $R^i{}_{jkl}$, a mixed tensor of Rank 4.

Appendix

A0. Why Derive Everything?

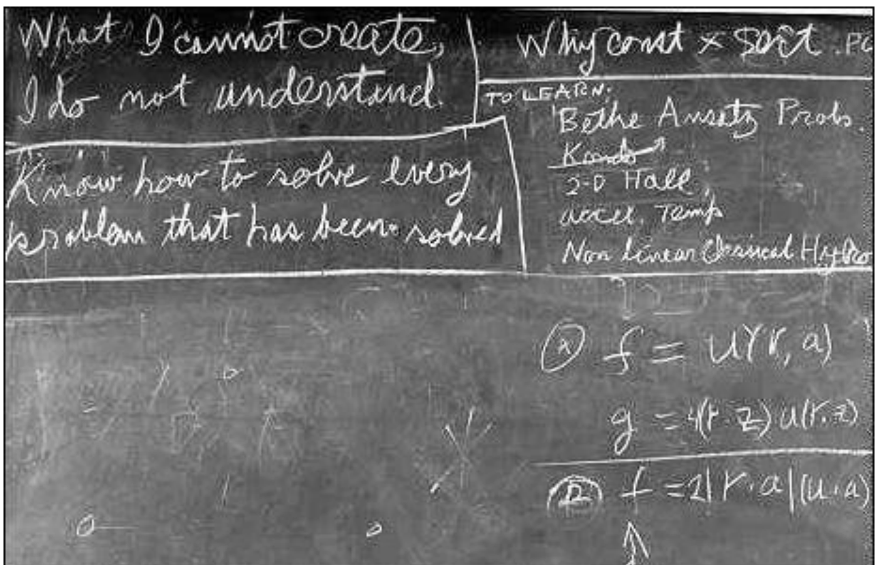


Photos (1947, 1961) by O. J. Turner of Princeton

Both special and general relativity start with an elegant foundation and everything is derived from these first principles. When I started my study of general relativity as an undergraduate, I liked how the theory is developed from Einstein's core ideas. Einstein is pictured at the far left (Princeton, 1947).

Later in graduate school I was fortunate to work under Dr. Y. S. Kim (right of Einstein) who formulated a quark model using relativistic harmonic oscillators. Every calculation could be done from scratch and I could even derive all the integral-table results I had to use.

I have always felt it important to derive everything from scratch. I was pleased to learn that Feynman had a similar attitude. See his blackboard below at the time of his death.



Richard Feynman

If Feynman couldn't derive it, he felt he did not understand it. So in this course, we aim to derive everything, starting with some trig derivations as a review of your calculus courses.

Note that you should have one year of calculus and one year of calculus-based physics as prerequisites to follow our course in general relativity.

A-1. Vector Review.

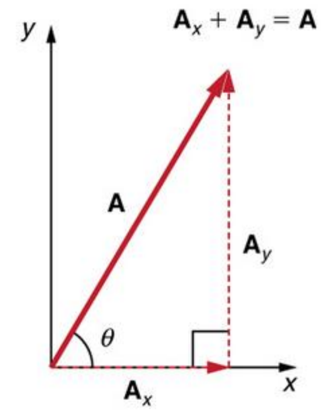


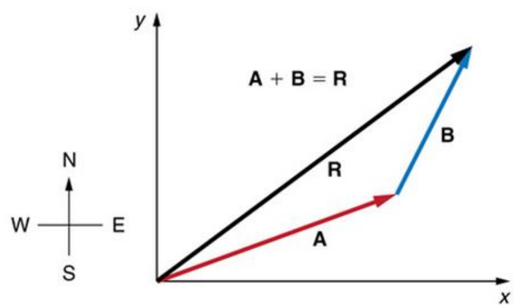
Figure Courtesy OpenStax College. Vector Addition and Subtraction: Analytical Methods, Connexions Website. http://cnx.org/content/m42128/1.10/, June 20, 2012.

$$\vec{A} = A_x \hat{i} + A_y \hat{j}$$

$$\vec{A}_x = A_x \hat{i} \quad \text{and} \quad \vec{A}_y = A_y \hat{j}$$

A-2. Vector Addition and Subtraction.

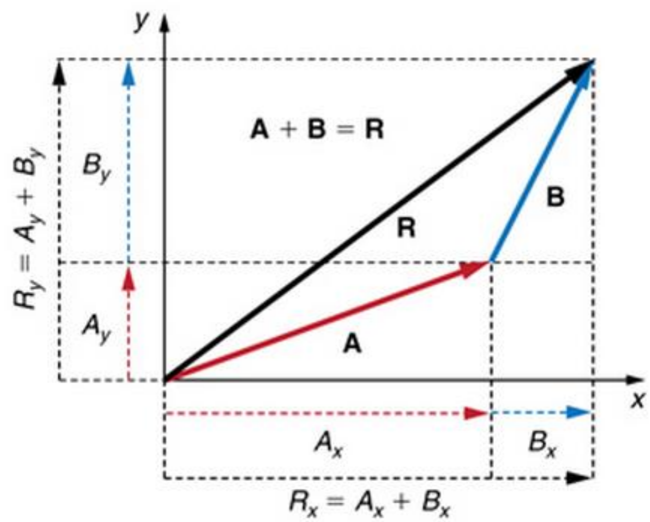
Figures Courtesy OpenStax College

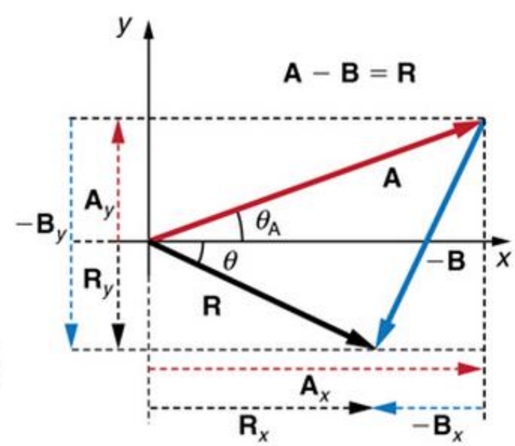


$$\vec{A} = A_x \hat{i} + A_y \hat{j}$$

$$\vec{B} = B_x \hat{i} + B_y \hat{j}$$

$$\vec{R} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j}$$





The Resultant: $\vec{R} = \vec{A} + \vec{B} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} = R_x \hat{i} + R_y \hat{j}$

Adding the Negative Vector (Subtraction): $\vec{A} - \vec{B} = (A_x - B_x)\hat{i} + (A_y - B_y)\hat{j}$

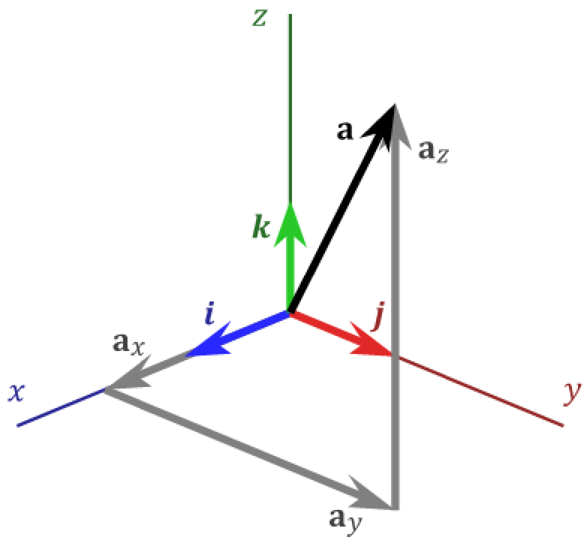


Image Courtesy Acdx, Wikipedia

Examples of vectors in three dimensions.

$$\vec{a} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$$

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$$

A-3. Scalar Multiplication.

$$\alpha \vec{A} = \alpha (A_x \hat{i} + A_y \hat{j} + A_z \hat{k})$$

$$\alpha \vec{A} = \alpha A_x \hat{i} + \alpha A_y \hat{j} + \alpha A_z \hat{k}$$

A-4. Dot Product.

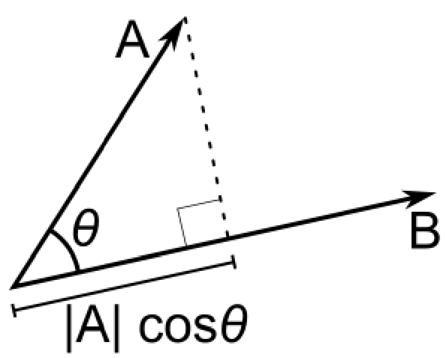


Image from Wikimedia Commons

$$\vec{A} \cdot \vec{B} = AB \cos\theta$$

Note: $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = (1)(1)\cos 0° = 1$

$$\hat{i} \cdot \hat{j} = \hat{i} \cdot \hat{k} = \hat{j} \cdot \hat{k} = (1)(1)\cos 90° = 0$$

Also note

$$\vec{B} \cdot \vec{A} = BA\cos\theta = AB\cos\theta = \vec{A} \cdot \vec{B}.$$

Using the rules for the dot product of the unit vectors we arrive at the following.

$$\vec{A}\cdot\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

$$\vec{A}\cdot\vec{B} = A_xB_x\,\hat{i}\cdot\hat{i} + A_xB_y\,\hat{i}\cdot\hat{j} + A_xB_z\,\hat{i}\cdot\hat{k}$$

$$+A_yB_x\,\hat{j}\cdot\hat{i} + A_yB_y\,\hat{j}\cdot\hat{j} + A_yB_z\,\hat{j}\cdot\hat{k}$$

$$+A_zB_x\,\hat{k}\cdot\hat{i} + A_zB_y\,\hat{k}\cdot\hat{j} + A_zB_z\,\hat{k}\cdot\hat{k}$$

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

More notation: $\hat{e}_1 = \hat{i}$, $\hat{e}_2 = \hat{j}$, and $\hat{e}_3 = \hat{k}$. Then $\vec{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3$.

The unit vectors are also called basis vectors and we use subscripts that can take on values 1, 2, and 3. The dot product of two arbitrary unit vectors can then be written as

$\hat{e}_i\cdot\hat{e}_j = \delta_{ij}$ where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

Here is a summation notation: $\vec{A} = \sum_{i=1}^{3} A_i\hat{e}_i$ and $\vec{B} = \sum_{i=1}^{3} B_i\hat{e}_i$

$$\vec{A}\cdot\vec{B} = \sum_{i=1}^{3} A_i\hat{e}_i\cdot\sum_{j=1}^{3} B_j\hat{e}_j = \sum_{i=1}^{3}\sum_{j=1}^{3} A_iB_j\,\hat{e}_i\cdot\hat{e}_j = \sum_{i=1}^{3}\sum_{j=1}^{3} A_iB_j\delta_{ij} = \sum_{i=1}^{3} A_iB_i$$

$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$ or $\vec{A}\cdot\vec{B} = A_1B_1 + A_2B_2 + A_3B_3$

Einstein Summation Convention: $\vec{A} = A_i\hat{e}_i$ and $\vec{B} = B_i\hat{e}_i$

$$\vec{A}\cdot\vec{B} = A_i\,\hat{e}_i\cdot B_j\,\hat{e}_j = A_i B_j\,\hat{e}_i\cdot\hat{e}_j = A_i B_j \delta_{ij} = A_i B_i$$

Leopold Kronecker (1823-1891)
Courtesy School of Mathematics and Statistics
University of St. Andrews, Scotland

The Kronecker Delta symbol is defined as

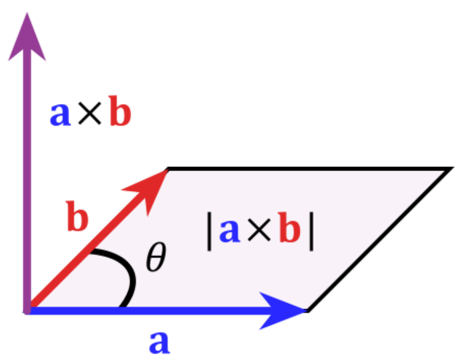
$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

and named after the German mathematician Leopold Kronecker. It is a symmetric symbol.

HWA-1 Angle Between Two Vectors Method 1. Find the angle theta between the two vectors using the two dot product definitions. Check your answer with a graphical diagram.

$$\vec{A} = \hat{i} - \hat{j} \text{ and } \vec{B} = \hat{i} + \hat{j}\,.$$

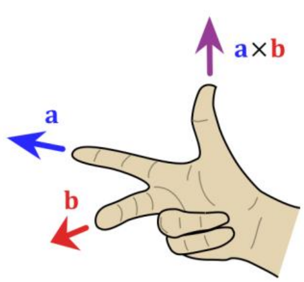
A-5. Cross Product.



Cross Product (Images Courtesy Acdx, Wikipedia)

$$\vec{A}\times\vec{B} = AB\sin\theta\,\hat{n}\;,\;\vec{a}\times\vec{b} = ab\sin\theta\,\hat{n}$$

where the unit vector $\hat{n}$ is perpendicular to the plane formed by $\vec{a}$ and $\vec{b}$, according to the right-hand rule as shown in the lower figure.



Or you can use the "right-hand screwdriver rule" where you get under the plane and apply the screwdriver to turn "a" into "b" advancing along "n". By the way ab sinθ is the area shown in the parallelogram.

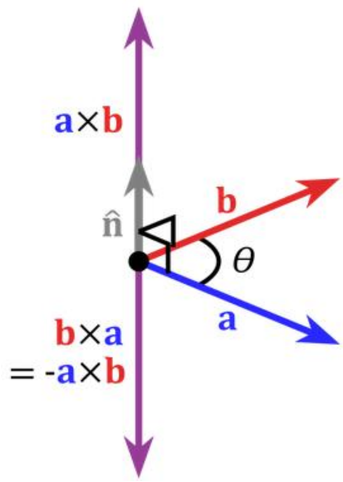


Image Courtesy Acdx, Wikipedia

Note that if you flip the order of the vectors, you get a vector in the opposite direction according to the right-hand rule.

$$\vec{b}\times\vec{a}=ba\sin\theta(-\hat{n})$$

$$\vec{b}\times\vec{a}=-\vec{a}\times\vec{b} \quad \text{and} \quad \vec{B}\times\vec{A}=-\vec{A}\times\vec{B}$$

The right hand-rule with the unit vectors gives us these relations below.

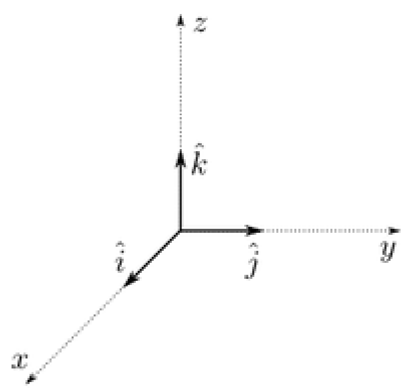


Image Courtesy Acdx, Wikipedia

$$\hat{i}\times\hat{j}=\hat{k} \qquad \hat{j}\times\hat{i}=-\hat{k} \qquad \hat{i}\times\hat{i}=0$$

$$\hat{j}\times\hat{k}=\hat{i} \qquad \hat{k}\times\hat{j}=-\hat{i} \qquad \hat{j}\times\hat{j}=0$$

$$\hat{k}\times\hat{i}=\hat{j} \qquad \hat{i}\times\hat{k}=-\hat{j} \qquad \hat{k}\times\hat{k}=0$$

We now apply these rules.

$$\vec{A}\times\vec{B}=(A_x\hat{i}+A_y\hat{j}+A_z\hat{k})\times(B_x\hat{i}+B_y\hat{j}+B_z\hat{k})$$ works out to:

$$\vec{A}\times\vec{B}=A_xB_x\,\hat{i}\times\hat{i}+A_xB_y\,\hat{i}\times\hat{j}+A_xB_z\,\hat{i}\times\hat{k}$$

$$+A_yB_x\,\hat{j}\times\hat{i}+A_yB_y\,\hat{j}\times\hat{j}+A_yB_z\,\hat{j}\times\hat{k}$$

$$+A_zB_x\,\hat{k}\times\hat{i}+A_zB_y\,\hat{k}\times\hat{j}+A_zB_z\,\hat{k}\times\hat{k}$$

$$\vec{A}\times\vec{B}=A_xB_x\cdot 0+A_xB_y\,\hat{k}+A_xB_z(-\hat{j})$$

$$+A_yB_x(-\hat{k})+A_yB_y\cdot 0+A_yB_z\,\hat{i}$$

$$+A_zB_x\,\hat{j}+A_zB_y(-\hat{i})+A_zB_z\cdot 0$$

$$\vec{A}\times\vec{B}=(A_yB_z-A_zB_y)\hat{i}+(A_zB_x-A_xB_z)\hat{j}+(A_xB_y-A_yB_x)\hat{k}$$

$$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}$$

$$=\hat{i}(A_yB_z-A_zB_y)-\hat{j}(A_xB_z-A_zB_x)+\hat{k}(A_xB_y-A_yB_x)$$

We now switch to our index notation. where $\hat{e}_1=\hat{i}$, $\hat{e}_2=\hat{j}$, and $\hat{e}_3=\hat{k}$.

The cross-product rules can be summarized by writing

$$\hat{e}_i\times\hat{e}_j=\varepsilon_{ijk}\,\hat{e}_k \text{ where}$$

$$\varepsilon_{ijk}=\begin{cases}+1 & \text{if } (i,j,k) \text{ is } (1,2,3),(3,1,2) \text{ or } (2,3,1),\\ -1 & \text{if } (i,j,k) \text{ is } (1,3,2),(3,2,1) \text{ or } (2,1,3),\\ 0 & \text{if } i=j \text{ or } j=k \text{ or } k=i\end{cases}$$

Tullio Levi-Civita (1873-1941)
Courtesy School of Mathematics and Statistics
University of St. Andrews, Scotland

The symbol ε_{ijk} is called the Levi-Civita or permutation symbol. It is an antisymmetric symbol. If you swap any two indices you introduce a minus sign. If any two indices are the same you get zero.

$$\vec{A}\times\vec{B}=\sum_{i=1}^{3}A_i\,\hat{e}_i\times\sum_{j=1}^{3}B_j\,\hat{e}_j=\sum_{i=1}^{3}\sum_{j=1}^{3}A_iB_j\,\hat{e}_i\times\hat{e}_j$$

$$\vec{A}\times\vec{B}=\sum_{i=1}^{3}\sum_{j=1}^{3}A_iB_j\varepsilon_{ijk}\,\hat{e}_k$$

The same with Einstein's summation convention is:

$$\vec{A} \times \vec{B} = A_i \hat{e}_i \times B_j \hat{e}_j = A_i B_j \varepsilon_{ijk} \hat{e}_k$$

HWA-2. Angle Between Vectors Method 2. Find the angle theta between the two vectors using the two cross product definitions. Check your answer against HWA-1.

$$\vec{A} = \hat{i} - \hat{j} \text{ and } \vec{B} = \hat{i} + \hat{j}.$$

A-6. Basic Trig Derivatives

HWA-3. Derivatives of tan x and cot x. Use the equations

$$\tan x = \frac{\sin x}{\cos x},\ \cot x = \frac{\cos x}{\sin x},\ \sec x = \frac{1}{\cos x},\ \csc x = \frac{1}{\sin x},$$

$$\frac{d \sin x}{dx} = \cos x,\ \frac{d \cos x}{dx} = -\sin x,$$

and the product rule $\frac{d(fg)}{dx} = \frac{df}{dx} g + f \frac{dg}{dx}$ where $f = \sin x$, $g = \frac{1}{\cos x}$ for the tangent case and $f = \cos x$, $g = \frac{1}{\sin x}$ for the cotangent case, to show that

$$\frac{d \tan x}{dx} = \sec^2 x \quad \text{and} \quad \frac{d \cot x}{dx} = -\csc^2 x.$$

A-7. Basic Derivatives of Inverse Trig Functions.

We will take the derivative of the inverse sine function first. Start with

$$\theta = \sin^{-1} x \equiv \arcsin x\,.$$

Then, we can write

$$\sin\theta = x \text{ and } \frac{d}{dx}\sin\theta = 1\,.$$

But $\frac{d}{dx}\sin\theta$ also equals $\frac{d\sin\theta}{d\theta}\frac{d\theta}{dx}$. Therefore,

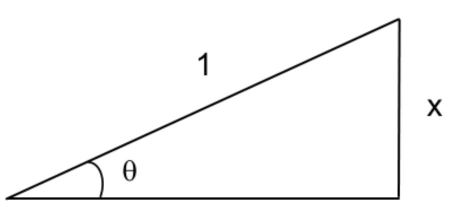
$$1 = \frac{d}{dx}\sin\theta = \frac{d\sin\theta}{d\theta}\frac{d\theta}{dx} = \cos\theta\frac{d\theta}{dx}\text{, i.e.,}$$

$$\cos\theta\frac{d\theta}{dx} = 1\,.$$

Substituting $\theta = \sin^{-1} x \equiv \arcsin x$,

$$\frac{d\sin^{-1} x}{dx} = \frac{1}{\cos\theta}\,.$$

The trick now is to get the right side in terms of x. The triangle below paves the way.



$$\sin\theta = x$$

$$\cos\theta = \sqrt{1 - x^2}$$

$$\frac{d\sin^{-1} x}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

HWA-4 Derivative of Inverse Sine. Consider the more general form $\sin^{-1}(x/a)$.

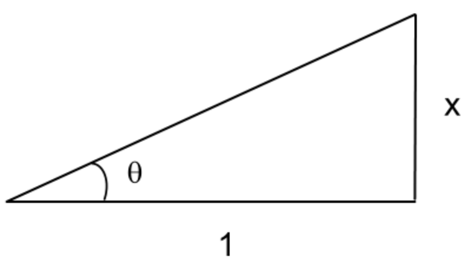
Show that $$\frac{d\sin^{-1}(x/a)}{dx} = \frac{1}{\sqrt{a^2 - x^2}}.$$

Hint: let $u = x/a$ and use $\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}$.

HWA-5 Derivative of Inverse Cosine. Start with $\cos\theta = x$ and show that

$$\frac{d\cos^{-1}x}{dx} = \frac{-1}{\sqrt{1-x^2}}$$ and then $$\frac{d\cos^{-1}(x/a)}{dx} = \frac{-1}{\sqrt{a^2 - x^2}}.$$

HWA-6 Derivative of Inverse Tangent. Start with $\tan\theta = x$ and the associated triangle below.



Show that

$$\frac{d\tan^{-1}x}{dx} = \frac{1}{1+x^2}.$$

Then show that

$$\frac{d\tan^{-1}(x/a)}{dx} = \frac{a}{a^2 + x^2}.$$

Concluding Remarks. So now you have derived the derivatives you used in earlier homework. If you like deriving everything, you have an affinity for theoretical physics. Feynman once said that a good theoretical physicist could derive anything more than one way. I think he said five ways or maybe it was even more. We will not go that far and be happy with one derivation for the most part.